An analytic function whose only singularities in the finite plane are poles is called a **meromorphic function.** Examples are rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$. ■

In this section we used Laurent series for investigating singularities. In the next section we shall use these series for an elegant integration method.

PROBLEM SET 16.2

1–10 ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \frac{1}{2}z$
2. $(z^4 - 81)^3$
3. $(z + 81i)^4$
4. $\tan^2 2z$
5. $z^{-2} \sin^2 \pi z$
6. $\cosh^4 z$
7. $z^4 + (1 - 8i)z^2 - 8i$
8. $(\sin z - 1)^3$
9. $\sin 2z \cos 2z$
10. $(z^2 - 8)^3(\exp(z^2) - 1)$
11. **Zeros.** If $f(z)$ is analytic and has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$ at z_0.
12. **TEAM PROJECT. Zeros. (a) Derivative.** Show that if $f(z)$ has a zero of order $n > 1$ at $z = z_0$, then $f'(z)$ has a zero of order $n - 1$ at z_0.

    (b) Poles and zeros. Prove Theorem 4.

    (c) Isolated *k*-points. Show that the points at which a nonconstant analytic function $f(z)$ has a given value k are isolated.

    (d) Identical functions. If $f_1(z)$ and $f_2(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.

13–22 SINGULARITIES

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

13. $\dfrac{1}{(z + 2i)^2} - \dfrac{z}{z - i} + \dfrac{z + 1}{(z - i)^2}$
14. $e^{z-i} + \dfrac{2}{z - i} - \dfrac{8}{(z - i)^3}$
15. $z \exp(1/(z - 1 - i)^2)$
16. $\tan \pi z$
17. $\cot^4 z$
18. $z^3 \exp\left(\dfrac{1}{z - 1}\right)$
19. $1/(e^z - e^{2z})$
20. $1/(\cos z - \sin z)$
21. $e^{1/(z-1)}/(e^z - 1)$
22. $(z - \pi)^{-1} \sin z$
23. **Essential singularity.** Discuss e^{1/z^2} in a similar way as $e^{1/z}$ is discussed in Example 3 of the text.
24. **Poles.** Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.
25. **Riemann sphere.** Assuming that we let the image of the x-axis be the meridians 0° and 180°, describe and sketch (or graph) the images of the following regions on the Riemann sphere: **(a)** $|z| > 100$, **(b)** the lower half-plane, **(c)** $\frac{1}{2} \leqq |z| \leqq 2$.

16.3 Residue Integration Method

We now cover a second method of evaluating complex integrals. Recall that we solved complex integrals directly by Cauchy's integral formula in Sec. 14.3. In Chapter 15 we learned about power series and especially Taylor series. We generalized Taylor series to Laurent series (Sec. 16.1) and investigated singularities and zeroes of various functions (Sec. 16.2). Our hard work has paid off and we see how much of the theoretical groundwork comes together in evaluating complex integrals by the residue method.

The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\, dz$$

taken around a simple closed path C. The idea is as follows.

If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem (Sec. 14.2), and we are done.

The situation changes if $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before. Then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$ (sometimes called a **deleted neighborhood**, an old-fashioned term that we shall not use). Now comes the key idea. The coefficient b_1 of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula (2) in Sec. 16.1 with $n = 1$, namely,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Now, since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients (see the examples in Sec. 16.1), we can find b_1 by one of those methods and then use the formula for b_1 for evaluating the integral, that is,

(1)
$$\oint_C f(z)\, dz = 2\pi i b_1.$$

Here we integrate counterclockwise around a simple closed path C that contains $z = z_0$ in its interior (but no other singular points of $f(z)$ on or inside C!).

The coefficient b_1 is called the **residue of** $f(z)$ at $z = z_0$ and we denote it by

(2)
$$b_1 = \operatorname*{Res}_{z=z_0} f(z).$$

EXAMPLE 1 **Evaluation of an Integral by Means of a Residue**

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

Solution. From (14) in Sec. 15.4 we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!} - \frac{z^3}{7!} + - \cdots$$

which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -\frac{1}{3}!$. From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4}\, dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

EXAMPLE 2 **CAUTION! Use the Right Laurent Series!**

Integrate $f(z) = 1/(z^3 - z^4)$ clockwise around the circle C: $|z| = \frac{1}{2}$.

Solution. $z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$. Now $z = 1$ lies outside C. Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$. This is series (I) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1).$$

We see from it that this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

CAUTION! Had we used the wrong series (II) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots \qquad (|z| > 1),$$

we would have obtained the wrong answer, 0, because this series has no power $1/z$. ■

Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

Simple Poles at z_0. A first formula for the residue at a simple pole is

(3)
$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z). \qquad \text{(Proof below).}$$

A second formula for the residue at a simple pole is

(4)
$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \qquad \text{(Proof below).}$$

In (4) we assume that $f(z) = p(z)/q(z)$ with $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, so that $f(z)$ has a simple pole at z_0 by Theorem 4 in Sec. 16.2.

PROOF We prove (3). For a simple pole at $z = z_0$ the Laurent series (1), Sec. 16.1, is

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R).$$

Here $b_1 \neq 0$. (Why?) Multiplying both sides by $z - z_0$ and then letting $z \to z_0$, we obtain the formula (3):

$$\lim_{z \to z_0} (z - z_0) f(z) = b_1 + \lim_{z \to z_0} (z - z_0)[a_0 + a_1(z - z_0) + \cdots] = b_1$$

where the last equality follows from continuity (Theorem 1, Sec. 15.3).

We prove (4). The Taylor series of $q(z)$ at a simple zero z_0 is

$$q(z) = (z - z_0) q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots.$$

Substituting this into $f = p/q$ and then f into (3) gives

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{(z - z_0) p(z)}{(z - z_0)[q'(z_0) + (z - z_0) q''(z_0)/2 + \cdots]}.$$

$z - z_0$ cancels. By continuity, the limit of the denominator is $q'(z_0)$ and (4) follows. ■

EXAMPLE 3 Residue at a Simple Pole

$f(z) = (9z + i)/(z^3 + z)$ has a simple pole at i because $z^2 + 1 = (z + i)(z - i)$, and (3) gives the residue

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z \to i} (z - i) \frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

By (4) with $p(i) = 9i + i$ and $q'(z) = 3z^2 + 1$ we confirm the result,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

Poles of Any Order at z_0. The residue of $f(z)$ at an mth-order pole at z_0 is

(5)
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}.$$

In particular, for a second-order pole ($m = 2$),

(5*)
$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \{[(z - z_0)^2 f(z)]'\}.$$

PROOF We prove (5). The Laurent series of $f(z)$ converging near z_0 (except at z_0 itself) is (Sec. 16.2)

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

where $b_m \neq 0$. The residue wanted is b_1. Multiplying both sides by $(z - z_0)^m$ gives

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_1(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots.$$

We see that b_1 is now the coefficient of the power $(z - z_0)^{m-1}$ of the power series of $g(z) = (z - z_0)^m f(z)$. Hence Taylor's theorem (Sec. 15.4) gives (5):

$$\begin{aligned} b_1 &= \frac{1}{(m-1)!} g^{(m-1)}(z_0) \\ &= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]. \end{aligned}$$

EXAMPLE 4 Residue at a Pole of Higher Order

$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$ has a pole of second order at $z = 1$ because the denominator equals $(z + 4)(z - 1)^2$ (verify!). From (5*) we obtain the residue

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{d}{dz} [(z - 1)^2 f(z)] = \lim_{z \to 1} \frac{d}{dz} \left(\frac{50z}{z + 4} \right) = \frac{200}{5^2} = 8.$$

Several Singularities Inside the Contour. Residue Theorem

Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour C. This is the purpose of the residue theorem. The extension is surprisingly simple.

THEOREM 1

Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_k$:

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{6}$$

PROOF We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated (Fig. 373 where $k = 3$). Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem we thus have

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz = 0, \tag{7}$$

the integral along C being taken *counterclockwise* and the other integrals *clockwise* (as in Figs. 354 and 355, Sec. 14.2). We take the integrals over $C_1, \cdots, C_k$ to the right and compensate the resulting minus sign by reversing the sense of integration. Thus,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz \tag{8}$$

where all the integrals are now taken counterclockwise. By (1) and (2),

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z), \qquad j = 1, \cdots, k,$$

so that (8) gives (6) and the residue theorem is proved. ■

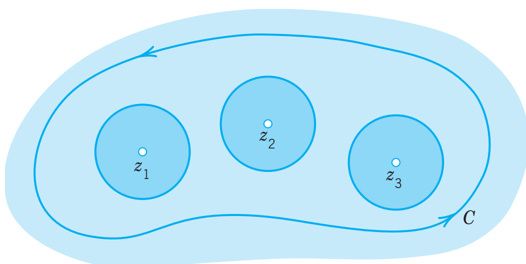

Fig. 373. Residue theorem

This important theorem has various applications in connection with complex and real integrals. Let us first consider some complex integrals. (Real integrals follow in the next section.)

EXAMPLE 5 Integration by the Residue Theorem. Several Contours

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}\,dz$$

Solution. The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

[Confirm this by (4).] *Answer:* (a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0. ■

EXAMPLE 6 Another Application of the Residue Theorem

Integrate $(\tan z)/(z^2-1)$ counterclockwise around the circle C: $|z| = \frac{3}{2}$.

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C. Because of the denominator $z^2 - 1 = (z-1)(z+1)$ the given function has simple poles at ± 1. We thus obtain from (4) and the residue theorem

$$\begin{aligned}\oint_C \frac{\tan z}{z^2-1}\,dz &= 2\pi i\left(\operatorname*{Res}_{z=1}\frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1}\frac{\tan z}{z^2-1}\right)\\ &= 2\pi i\left(\left.\frac{\tan z}{2z}\right|_{z=1} + \left.\frac{\tan z}{2z}\right|_{z=-1}\right)\\ &= 2\pi i \tan 1 = 9.7855i.\end{aligned}$$

■

EXAMPLE 7 Poles and Essential Singularities

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right)dz.$$

Solution. Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4); note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16},$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2}\cdot\frac{1}{z} + \cdots \qquad (|z| > 0).$$

Answer: $2\pi i(-\frac{1}{16} - \frac{1}{16} + \frac{1}{2}\pi^2) = \pi(\pi^2 - \frac{1}{4})i = 30.221i$ by the residue theorem. ■

PROBLEM SET 16.3

1. Verify the calculations in Example 3 and find the other residues.
2. Verify the calculations in Example 4 and find the other residue.

3–12 RESIDUES

Find all the singularities in the finite plane and the corresponding residues. Show the details.

3. $\dfrac{\sin 2z}{z^6}$
4. $\dfrac{\cos z}{z^4}$
5. $\dfrac{8}{1+z^2}$
6. $\tan z$
7. $\cot \pi z$
8. $\dfrac{\pi}{(z^2-1)^2}$
9. $\dfrac{1}{1-e^z}$
10. $\dfrac{z^4}{z^2-iz+2}$
11. $\dfrac{e^z}{(z-\pi i)^3}$
12. $e^{1/(1-z)}$
13. **CAS PROJECT. Residue at a Pole.** Write a program for calculating the residue at a pole of any order in the finite plane. Use it for solving Probs. 5–10.

14–25 RESIDUE INTEGRATION

Evaluate (counterclockwise). Show the details.

14. $\displaystyle\oint_C \frac{z-23}{z^2-4z-5}\,dz, \quad C: |z-2-i| = 3.2$
15. $\displaystyle\oint_C \tan 2\pi z\,dz, \quad C: |z-0.2| = 0.2$
16. $\displaystyle\oint_C e^{1/z}\,dz, \quad C$: the unit circle
17. $\displaystyle\oint_C \frac{e^z}{\cos z}\,dz, \quad C: |z-\pi i/2| = 4.5$
18. $\displaystyle\oint_C \frac{z+1}{z^4-2z^3}\,dz, \quad C: |z-1| = 2$
19. $\displaystyle\oint_C \frac{\sinh z}{2z-i}\,dz, \quad C: |z-2i| = 2$
20. $\displaystyle\oint_C \frac{dz}{(z^2+1)^3}, \quad C: |z-i| = 3$
21. $\displaystyle\oint_C \frac{\cos \pi z}{z^5}\,dz, \quad C: |z| = \tfrac{1}{2}$
22. $\displaystyle\oint_C \frac{z^2 \sin z}{4z^2-1}\,dz, \quad C$ the unit circle
23. $\displaystyle\oint_C \frac{30z^2-23z+5}{(2z-1)^2(3z-1)}\,dz, \quad C$ the unit circle
24. $\displaystyle\oint_C \frac{\exp(-z^2)}{\sin 4z}\,dz, \quad C: |z| = 1.5$
25. $\displaystyle\oint_C \frac{z \cosh \pi z}{z^4+13z^2+36}\,dz, \quad |z| = \pi$

16.4 Residue Integration of Real Integrals

Surprisingly, residue integration can also be used to evaluate certain classes of complicated real integrals. This shows an advantage of complex analysis over real analysis or calculus.

Integrals of Rational Functions of $\cos\theta$ and $\sin\theta$

We first consider integrals of the type

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \tag{1}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite (does not become infinite) on the interval of integration. Setting $e^{i\theta} = z$, we obtain

$$\textbf{(2)} \qquad \begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right). \end{aligned}$$

Since F is rational in $\cos\theta$ and $\sin\theta$, Eq. (2) shows that F is now a rational function of z, say, $f(z)$. Since $dz/d\theta = ie^{i\theta}$, we have $d\theta = dz/iz$ and the given integral takes the form

$$(3) \qquad J = \oint_C f(z)\,\frac{dz}{iz}$$

and, as θ ranges from 0 to 2π in (1), the variable $z = e^{i\theta}$ ranges counterclockwise once around the unit circle $|z| = 1$. (Review Sec. 13.5 if necessary.)

EXAMPLE 1 An Integral of the Type (1)

Show by the present method that $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi.$

Solution. We use $\cos\theta = \frac{1}{2}(z + 1/z)$ and $d\theta = dz/iz$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2} - \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)} = \oint_C \frac{dz}{-\dfrac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

We see that the integrand has a simple pole at $z_1 = \sqrt{2} + 1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2 = \sqrt{2} - 1$ (where $z - \sqrt{2} + 1 = 0$) inside C with residue [by (3), Sec. 16.3]

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left[\frac{1}{z - \sqrt{2} - 1}\right]_{z=\sqrt{2}-1}$$

$$= -\frac{1}{2}.$$

Answer: $2\pi i(-2/i)(-\frac{1}{2}) = 2\pi$. (Here $-2/i$ is the factor in front of the last integral.) ■

As another large class, let us consider real integrals of the form

$$(4) \qquad \int_{-\infty}^{\infty} f(x)\,dx.$$

Such an integral, whose interval of integration is not finite is called an **improper integral**, and it has the meaning

$$(5') \qquad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx.$$

If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \tag{5}$$

The limit in (5) is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx.$$

It may exist even if the limits in (5′) do not. *Example:*

$$\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{R^2}{2}\right) = 0, \qquad \text{but} \qquad \lim_{b\to\infty} \int_{0}^{b} x\,dx = \infty.$$

We assume that the function $f(x)$ in (4) is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of the numerator. Then the limits in (5′) exist, and we may start from (5). We consider the corresponding contour integral

$$\oint_C f(z)\,dz \tag{5*}$$

around a path C in Fig. 374. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose R large enough, then C encloses all these poles. By the residue theorem we then obtain

$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

where the sum consists of all the residues of $f(z)$ at the points in the upper half-plane at which $f(z)$ has a pole. From this we have

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)\,dz. \tag{6}$$

We prove that, if $R \to \infty$, the value of the integral over the semicircle S approaches zero. If we set $z = Re^{i\theta}$, then S is represented by $R = \text{const}$, and as z ranges along S, the variable θ ranges from 0 to π. Since, by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

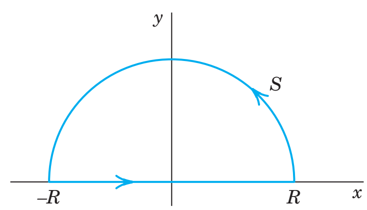

Fig. 374. Path C of the contour integral in (5*)

for sufficiently large constants k and R_0. By the ML-inequality in Sec. 14.1,

$$\left| \int_S f(z)\, dz \right| < \frac{k}{R^2} \pi R = \frac{k\pi}{R} \qquad (R > R_0).$$

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

(7) $$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z)$$

where we sum over all the residues of $f(z)$ at the poles of $f(z)$ in the upper half-plane.

EXAMPLE 2 An Improper Integral from 0 to ∞

Using (7), show that

$$\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$

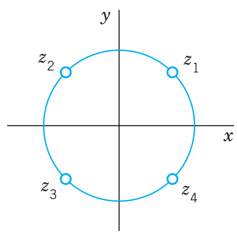

Fig. 375. Example 2

Solution. Indeed, $f(z) = 1/(1 + z^4)$ has four simple poles at the points (make a sketch)

$$z_1 = e^{\pi i/4}, \qquad z_2 = e^{3\pi i/4}, \qquad z_3 = e^{-3\pi i/4}, \qquad z_4 = e^{-\pi i/4}.$$

The first two of these poles lie in the upper half-plane (Fig. 375). From (4) in the last section we find the residues

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1 + z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4} e^{-3\pi i/4} = -\frac{1}{4} e^{\pi i/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1 + z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4} e^{-9\pi i/4} = \frac{1}{4} e^{-\pi i/4}.$$

(Here we used $e^{\pi i} = -1$ and $e^{-2\pi i} = 1$.) By (1) in Sec. 13.6 and (7) in this section,

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = -\frac{2\pi i}{4}(e^{\pi i/4} - e^{-\pi i/4}) = -\frac{2\pi i}{4} \cdot 2i \sin\frac{\pi}{4} = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since $1/(1 + x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^\infty \frac{dx}{1 + x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$ ■

Fourier Integrals

The method of evaluating (4) by creating a closed contour (Fig. 374) and "blowing it up" extends to integrals

$$\text{(8)} \qquad \int_{-\infty}^{\infty} f(x)\cos sx\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\sin sx\, dx \qquad (s \text{ real})$$

as they occur in connection with the Fourier integral (Sec. 11.7).

If $f(x)$ is a rational function satisfying the assumption on the degree as for (4), we may consider the corresponding integral

$$\oint_C f(z)e^{isz}\, dz \qquad (s \text{ real and positive})$$

over the contour C in Fig. 374. Instead of (7) we now get

$$\text{(9)} \qquad \int_{-\infty}^{\infty} f(x)e^{isx}\, dx = 2\pi i \sum \operatorname{Res}\,[f(z)e^{isz}] \qquad (s > 0)$$

where we sum the residues of $f(z)e^{isz}$ at its poles in the upper half-plane. Equating the real and the imaginary parts on both sides of (9), we have

$$\textbf{(10)} \qquad \begin{aligned} \int_{-\infty}^{\infty} f(x)\cos sx\, dx &= -2\pi \sum \operatorname{Im}\operatorname{Res}\,[f(z)e^{isz}], \\ \int_{-\infty}^{\infty} f(x)\sin sx\, dx &= 2\pi \sum \operatorname{Re}\operatorname{Res}\,[f(z)e^{isz}]. \end{aligned} \qquad (s > 0)$$

To establish (9), we must show [as for (4)] that the value of the integral over the semicircle S in Fig. 374 approaches 0 as $R \to \infty$. Now $s > 0$ and S lies in the upper half-plane $y \geqq 0$. Hence

$$|e^{isz}| = |e^{is(x+iy)}| = |e^{isx}||e^{-sy}| = 1 \cdot e^{-sy} \leqq 1 \qquad (s > 0, \quad y \geqq 0).$$

From this we obtain the inequality $|f(z)e^{isz}| = |f(z)||e^{isz}| \leqq |f(z)| \quad (s > 0, \quad y \geqq 0)$. This reduces our present problem to that for (4). Continuing as before gives (9) and (10). ■

EXAMPLE 3 An Application of (10)

Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2}\, dx = \frac{\pi}{k} e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2}\, dx = 0 \qquad (s > 0, k > 0).$$

Solution. In fact, $e^{isz}/(k^2 + z^2)$ has only one pole in the upper half-plane, namely, a simple pole at $z = ik$, and from (4) in Sec. 16.3 we obtain

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \frac{e^{-ks}}{2ik}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2 + x^2}\, dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{-ks}.$$

Since $e^{isx} = \cos sx + i \sin sx$, this yields the above results [see also (15) in Sec. 11.7.]

Another Kind of Improper Integral

We consider an improper integral

$$\int_A^B f(x)\, dx \tag{11}$$

whose integrand becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty.$$

By definition, this integral (11) means

$$\int_A^B f(x)\, dx = \lim_{\epsilon \to 0} \int_A^{a-\epsilon} f(x)\, dx + \lim_{\eta \to 0} \int_{a+\eta}^{B} f(x)\, dx \tag{12}$$

where both ϵ and η approach zero independently and through positive values. It may happen that neither of these two limits exists if ϵ and η go to 0 independently, but the limit

$$\lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\, dx + \int_{a+\epsilon}^{B} f(x)\, dx\right] \tag{13}$$

exists. This is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_A^B f(x)\, dx.$$

For example,

$$\text{pr. v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\epsilon \to 0} \left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3}\right] = 0;$$

the principal value exists, although the integral itself has no meaning.

In the case of simple poles on the real axis we shall obtain a formula for the principal value of an integral from $-\infty$ to ∞. This formula will result from the following theorem.

THEOREM 1

Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then (Fig. 376)

$$\lim_{r \to 0} \int_{C_2} f(z)\, dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

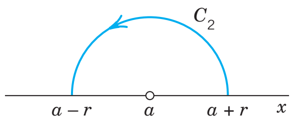

Fig. 376. Theorem 1

PROOF By the definition of a simple pole (Sec. 16.2) the integrand $f(z)$ has for $0 < |z - a| < R$ the Laurent series

$$f(z) = \frac{b_1}{z - a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z).$$

Here $g(z)$ is analytic on the semicircle of integration (Fig. 376)

$$C_2\colon \quad z = a + re^{i\theta}, \qquad 0 \leqq \theta \leqq \pi$$

and for all z between C_2 and the x-axis, and thus bounded on C_2, say, $|g(z)| \leqq M$. By integration,

$$\int_{C_2} f(z)\, dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}}\, ire^{i\theta}\, d\theta + \int_{C_2} g(z)\, dz = b_1 \pi i + \int_{C_2} g(z)\, dz.$$

The second integral on the right cannot exceed $M\pi r$ in absolute value, by the ML-inequality (Sec. 14.1), and $ML = M\pi r \to 0$ as $r \to 0$. ■

Figure 377 shows the idea of applying Theorem 1 to obtain the principal value of the integral of a rational function $f(x)$ from $-\infty$ to ∞. For sufficiently large R the integral over the entire contour in Fig. 377 has the value J given by $2\pi i$ times the sum of the residues of $f(z)$ at the singularities in the upper half-plane. We assume that $f(x)$ satisfies the degree condition imposed in connection with (4). Then the value of the integral over the large

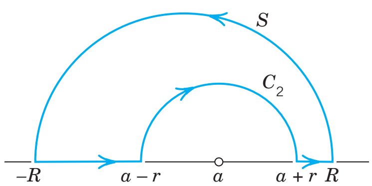

Fig. 377. Application of Theorem 1

semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

by Theorem 1. Together this shows that the principal value P of the integral from $-\infty$ to ∞ plus K equals J; hence $P = J - K = J + \pi i \operatorname{Res}_{z=a} f(z)$. If $f(z)$ has several simple poles on the real axis, then K will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

$$\text{(14)} \qquad \text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

EXAMPLE 4 Poles on the Real Axis

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i \left(\frac{3-i}{20}\right) + \pi i \left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10}.$$

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.

PROBLEM SET 16.4

1–9 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals and show the details of your work.

1. $\int_0^{\pi} \frac{2\,d\theta}{k - \cos\theta}$

2. $\int_0^{\pi} \frac{d\theta}{\pi + 3\cos\theta}$

3. $\int_0^{2\pi} \frac{1 + \sin\theta}{3 + \cos\theta}\,d\theta$

4. $\int_0^{2\pi} \frac{1 + 4\cos\theta}{17 - 8\cos\theta}\,d\theta$

5. $\int_0^{2\pi} \frac{\cos^2\theta}{5 - 4\cos\theta}\,d\theta$

6. $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$

7. $\int_0^{2\pi} \frac{a}{a - \sin\theta}\,d\theta$

8. $\int_0^{2\pi} \frac{1}{8 - 2\sin\theta}\,d\theta$

9. $\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\,d\theta$

10–22 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your work.

10. $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3}$

11. $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}$

12. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2}$

13. $\int_{-\infty}^{\infty} \frac{x}{(x^2 + 1)(x^2 + 4)}\,dx$

14. $\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1}\,dx$

15. $\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1}\,dx$

16. $\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2 + 1)^2}\,dx$

17. $\int_{-\infty}^{\infty} \frac{\sin 3x}{x^4 + 1}\,dx$

18. $\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4 + 5x^2 + 4}\,dx$

19. $\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$

20. $\int_{-\infty}^{\infty} \frac{x}{8 - x^3}\,dx$

21. $\int_{-\infty}^{\infty} \frac{\sin x}{(x - 1)(x^2 + 4)}\,dx$

22. $\int_{-\infty}^{\infty} \frac{dx}{x^2 - ix}$

23–26 IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

23. $\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$

24. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$

25. $\int_{-\infty}^{\infty} \frac{x + 5}{x^3 - x}\,dx$

26. $\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\,dx$

27. CAS EXPERIMENT. Simple Poles on the Real Axis. Experiment with integrals $\int_{-\infty}^{\infty} f(x)\,dx$, $f(x) = [(x - a_1)(x - a_2)\cdots(x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.

28. TEAM PROJECT. Comments on Real Integrals.

(a) Formula (10) follows from (9). Give the details.

(b) Use of auxiliary results. Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a$, a, $a + ib$, $-a + ib$, letting $a \to \infty$, and using

$$\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

show that

$$\int_0^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}.$$

(This integral is needed in heat conduction in Sec. 12.7.)

(c) Inspection. Solve Probs. 13 and 17 without calculation.

CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS

1. What is a Laurent series? Its principal part? Its use? Give simple examples.
2. What kind of singularities did we discuss? Give definitions and examples.
3. What is the residue? Its role in integration? Explain methods to obtain it.
4. Can the residue at a singularity be zero? At a simple pole? Give reason.
5. State the residue theorem and the idea of its proof from memory.
6. How did we evaluate real integrals by residue integration? How did we obtain the closed paths needed?

7. What are improper integrals? Their principal value? Why did they occur in this chapter?

8. What do you know about zeros of analytic functions? Give examples.

9. What is the extended complex plane? The Riemann sphere R? Sketch $z = 1 + i$ on R.

10. What is an entire function? Can it be analytic at infinity? Explain the definitions.

11–18 COMPLEX INTEGRALS

Integrate counterclockwise around C. Show the details.

11. $\dfrac{\sin 3z}{z^2}$, $\quad C: |z| = \pi$

12. $e^{2/z}$, $\quad C: |z - 1 - i| = 2$

13. $\dfrac{5z^3}{z^2 + 4}$, $\quad C: |z| = 3$

14. $\dfrac{5z^3}{z^2 + 4}$, $\quad C: |z - i| = \pi i/2$

15. $\dfrac{25z^2}{(z - 5)^2}$, $\quad C: |z - 5| = 1$

16. $\dfrac{15z + 9}{z^3 - 9z}$, $\quad C: |z| = 4$

17. $\dfrac{\cos z}{z^n}$, $\quad n = 0, 1, 2, \cdots, \quad C: |z| = 1$

18. $\cot 4z$, $\quad C: |z| = \frac{3}{4}$

19–25 REAL INTEGRALS

Evaluate by the methods of this chapter. Show details.

19. $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 - 5 \sin \theta}$

20. $\displaystyle\int_0^{2\pi} \frac{\sin \theta}{3 + \cos \theta}\, d\theta$

21. $\displaystyle\int_0^{2\pi} \frac{\sin \theta}{34 - 16 \sin \theta}\, d\theta$

22. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + 4x^4}$

23. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(1 + x^2)^2}\, dx$

24. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 4ix}$

25. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1}\, dx$

SUMMARY OF CHAPTER 16
Laurent Series. Residue Integration

A **Laurent series** is a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{1}$$ (Sec. 16.1)

or, more briefly written [but this means the same as (1)!]

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^* \tag{1*}$$

where $n = 0, \pm 1, \pm 2, \cdots$. This series converges in an open annulus (ring) A with center z_0. In A the function $f(z)$ is analytic. At points not in A it may have singularities. The first series in (1) is a power series. In a given annulus, a Laurent series of $f(z)$ is unique, but $f(z)$ may have different Laurent series in different annuli with the same center.

Of particular importance is the Laurent series (1) that converges in a neighborhood of z_0 except at z_0 itself, say, for $0 < |z - z_0| < R$ ($R > 0$, suitable). The series (or finite sum) of the negative powers in this Laurent series is called the **principal part** of $f(z)$ at z_0. The coefficient b_1 of $1/(z - z_0)$ in this series is called the **residue** of $f(z)$ at z_0 and is given by [see (1) and (1*)]

$$(2) \quad b_1 = \operatorname*{Res}_{z \to z_0} f(z) = \frac{1}{2\pi i} \oint_C f(z^*)\, dz^*. \qquad \text{Thus} \qquad \oint_C f(z^*)\, dz^* = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

b_1 can be used for ***integration*** as shown in (2) because it can be found from

$$(3) \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right), \qquad \text{(Sec. 16.3)},$$

provided $f(z)$ has at z_0 a **pole of order *m***; by definition this means that principal part has $1/(z - z_0)^m$ as its highest negative power. Thus for a simple pole ($m = 1$),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z); \qquad \text{also,} \qquad \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

If the principal part is an infinite series, the singularity of $f(z)$ at z_0 is called an **essential singularity** (Sec. 16.2).

Section 16.2 also discusses the *extended complex plane*, that is, the complex plane with an improper point ∞ ("infinity") attached.

Residue integration may also be used to evaluate certain classes of complicated real integrals (Sec. 16.4).