Models of LTI Systems

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Introduction

A model of a system is a description of (some of) its properties, suitable for a certain purpose.

System Identification

Constructing or Selecting Models

for certain purpose
Linear Models & Sets of LM
Specified by

\[\{g(k)\}_{1}^{\infty}\]  \[\Phi_v(\omega) = \lambda |H(e^{i\omega})|^2\]

\[u(t) \rightarrow \text{Impulse Response} \rightarrow \text{Impulse Response} \rightarrow y(t)\]

\[e(t) \text{ White noise}\]

\[y(t) = G(q) \cdot u(t) + H(q) \cdot e(t)\]

\[G(q) = \sum_{i=1}^{\infty} g(k)q^{-k}\]  \[H(q) = 1 + \sum_{i=1}^{\infty} h(k)q^{-k}\]
Linear Models & Sets of LM (3)

\[ y(t) = G(q) \ u(t) + H(q) \ e(t) \]

Parameters to be determined

\[ y(t) = G(q, \theta) \ u(t) + H(q, \theta) \ e(t) \]

No longer “a model” but “a set of models”

Linear Models & Sets of LM (4)

Use one-step-ahead prediction to emphasize its dependence on \( \theta \)

\[ \hat{y}(t|\theta) = H^{-1}(q, \theta) \ G(q, \theta) \ u(t) + [1 - H^{-1}(q, \theta)] \ y(t) \]

(4.6)

In this book we have three different ways to describe the function in term of \( \theta \)
Family of Transfer Functions

Input-output relationship described by Linear Difference Equation

Equation Error Model Structure

\[ y(t) + a_1 y(t - 1) + \cdots + a_{n_a} y(t - n_a) = b_1 u(t - 1) + \cdots + b_{n_b} u(t - n_b) + e(t) \]

\[ \theta = [a_1 \ a_2 \cdots a_{n_a} \ b_1 \cdots b_{n_b}]^T \]

ARX Model

Family of Transfer Functions (2)

defined

\[ A(q) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a} \]
\[ B(q) = b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b} \]

\[ y(t) + a_1 y(t - 1) + \cdots + a_{n_a} y(t - n_a) = b_1 u(t - 1) + \cdots + b_{n_b} u(t - n_b) + e(t) \]

\[ [1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}] y(t) = [b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}] u(t) + e(t) \]

\[ A(q) y(t) = B(q) u(t) + e(t) \]
Family of Transfer Functions (3)

\[ y(t) = G(q, \theta) u(t) + H(q, \theta) e(t) \]

\[ A(q) y(t) = B(q) u(t) + e(t) \]

\[ y(t) = \frac{B(q)}{A(q)} u(t) + \frac{1}{A(q)} e(t) \]

\[ G(q, \theta) \quad H(q, \theta) \]

Family of Transfer Functions (4)

Equation Error Model Structure  ARX Model

\[ y(t) = \frac{B(q)}{A(q)} u(t) + \frac{1}{A(q)} e(t) \]

\[ A(q) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a} \]

\[ B(q) = b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b} \]

\[ \theta = [a_1 \quad a_2 \cdots a_{n_a} \quad b_1 \cdots b_{n_b}]^T \]

\[ G(q, \theta) = \frac{B(q)}{A(q)} \quad H(q, \theta) = \frac{1}{A(q)} \]
**Family of Transfer Functions (5)**

one-step-ahead prediction
\[
\hat{y}(t|\theta) = H^{-1}(q, \theta) G(q, \theta) u(t) + \left[ 1 - H^{-1}(q, \theta) \right] y(t)
\]

\[
G(q, \theta) = \frac{B(q)}{A(q)} \quad H(q, \theta) = \frac{1}{A(q)} \quad H^{-1}(q, \theta) = A(q)
\]

\[
\hat{y}(t|\theta) = B(q) u(t) + \left[ 1 - A(q) \right] y(t)
\]

**Family of Transfer Functions (6)**

**ARMAX Model Structure**

Basic disadvantage of ARX Model is the lack of freedom in describing the properties of disturbance

\[
y(t) + a_1 y(t-1) + \cdots + a_{n_a} y(t-n_a) = b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) + e(t) + c_1 e(t-1) + \cdots + c_{n_c} e(t-n_c)
\]

\[
C(q) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}
\]

\[
A(q) y(t) = B(q) u(t) + C(q) e(t)
\]
Family of Transfer Functions (7)

\[ A(q)y(t) = B(q)u(t) + C(q)e(t) \]

\[ y(t) = \frac{B(q)}{A(q)}u(t) + \frac{C(q)}{A(q)}e(t) \]

\[ G(q, \theta) \quad H(q, \theta) \]

\[ \theta = [a_1 \ a_2 \cdots a_{na} \ b_1 \cdots b_{nb} \ c_1 \cdots c_{nc}]^T \]

Family of Transfer Functions (8)

Pseudolinear Regressions

\[ \hat{y}(t|\theta) = H^{-1}(q, \theta) C(q, \theta) u(t) + [1 - H^{-1}(q, \theta)] y(t) \]

\[ G(q, \theta) = \frac{B(q)}{A(q)} \quad H(q, \theta) = \frac{C(q)}{A(q)} \quad H^{-1}(q, \theta) = \frac{A(q)}{C(q)} \]

\[ \hat{y}(t|\theta) = \frac{B(q)}{C(q)} u(t) + \left[ 1 - \frac{A(q)}{C(q)} \right] y(t) \]

*The prediction is obtained by filtering u and y through a filter with denominator dynamics determined by C(q)*
Family of Transfer Functions (9)

\[ \hat{y}(t|\theta) = \frac{B(q)}{C(q)} u(t) + \left[ 1 - \frac{A(q)}{C(q)} \right] y(t) \]

\[ \hat{y}(t|\theta) = \frac{B(q)}{C(q)} u(t) + \left[ \frac{C(q) - A(q)}{C(q)} \right] y(t) \]

\[ C(q) \hat{y}(t|\theta) = B(q) u(t) + [C(q) - A(q)] y(t) \]

\[ [1 - C(q)] \hat{y}(t|\theta) \]

\[ [1 - C(q)] \hat{y}(t|\theta) \]

\[ \hat{y}(t|\theta) = B(q) u(t) + [1 - A(q)] y(t) + [C(q) - 1] [y(t) - \hat{y}(t|\theta)] \]

(4.19)

Family of Transfer Functions (10)

Other Equation-Error-Type Structure

ARMAX Model

\[ A(q)y(t) = B(q)u(t) + C(q)e(t) \]

Error modeled as Moving Average process

\[ A(q)y(t) = B(q)u(t) + \frac{1}{D(q)} e(t) \]  

\[ D(q) = 1 + d_1 q^{-1} + \cdots + d_{nd} q^{-nd} \]

ARARX Model
Family of Transfer Functions (11)

Output Error (OE) Model Structure

\[ A(q)y(t) = B(q)u(t) + e(t) \]

Equation error model

\[ A(q)y(t) = B(q)u(t) + C(q)e(t) \]

\[ A(q)y(t) = B(q)u(t) + \frac{1}{D(q)}e(t) \]

The Transfer Function \( G \) and \( H \) have the polynomial \( A \)

Output = Undisturbed Output + Disturbance

\[ y(t) = w(t) + e(t) \]
\[ w(t) + f_1 w(t - 1) + \cdots + f_{n_f} w(t - n_f) = b_1 u(t - 1) + \cdots + b_{n_b} u(t - n_b) + \]

\[ F(q) = 1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f} \]

\[ F(q) w(t) = B(q) u(t) \]

\[ y(t) = w(t) + e(t) \]

\[ y(t) = \frac{B(q)}{F(q)} u(t) + e(t) \hspace{1cm} (4.25) \]

**Family of Transfer Functions (12)**

**Box-Jenkins Model Structure**

**Development of OE Model**

\[ y(t) = \frac{B(q)}{F(q)} u(t) + \frac{C(q)}{D(q)} e(t) \]

\[ G(q, \theta) \quad H(q, \theta) \]
General Family of Transfer Functions

General Family of Model Structures

\[ A(q)y(t) = \frac{B(q)}{F(q)} u(t) + \frac{C(q)}{D(q)} e(t) \] \hspace{1cm} (4.33)

<table>
<thead>
<tr>
<th>Polynomials Used</th>
<th>Name of Model Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>FIR (Finite Impulse Response)</td>
</tr>
<tr>
<td>A B</td>
<td>ARX</td>
</tr>
<tr>
<td>A B C</td>
<td>ARMAX</td>
</tr>
<tr>
<td>A C</td>
<td>ARMA</td>
</tr>
<tr>
<td>A B D</td>
<td>ARARX</td>
</tr>
<tr>
<td>A B C D</td>
<td>ARARMAX</td>
</tr>
<tr>
<td>B F</td>
<td>OE (Output-Error)</td>
</tr>
<tr>
<td>B F C D</td>
<td>BJ (Box-Jenkins)</td>
</tr>
</tbody>
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