Markov Chains

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The Bernoulli and Poisson processes studied in the preceding chapter are memoryless, in the sense that the future does not depend on the past: the occurrences of new “successes” or “arrivals” do not depend on the past history of the process. In this chapter, we consider processes where the future depends on and can be predicted to some extent by what has happened in the past.

We emphasize models where the effect of the past on the future is summarized by a state, which changes over time according to given probabilities. We restrict ourselves to models in which the state can only take a finite number values and can change according to probabilities that do not depend on the time of the change. We want to analyze the probabilistic properties of the sequence of state values.

The range of applications of the type of models described in this chapter is truly vast. It includes just about any dynamical system whose evolution over time involves uncertainty, provided the state of the system is suitably defined. Such systems arise in a broad variety of fields, such as, for example, communications, automatic control, signal processing, manufacturing, economics, and operations research.

7.1 DISCRETE-TIME MARKOV CHAINS

We will first consider discrete-time Markov chains, in which the state changes at certain discrete time instants, indexed by an integer variable \( n \). At each time step \( n \), the state of the chain is denoted by \( X_n \) and belongs to a finite set \( S \) of possible states, called the state space. Without loss of generality, and unless there is a statement to the contrary, we will assume that \( S = \{1, \ldots, m\} \), for some positive integer \( m \). The Markov chain is described in terms of its transition probabilities \( p_{ij} \): whenever the state happens to be \( i \), there is probability \( p_{ij} \) that the next state is equal to \( j \). Mathematically, 

\[
p_{ij} = \Pr(X_{n+1} = j \mid X_n = i), \quad i, j \in S.
\]

The key assumption underlying Markov chains is that the transition probabilities \( p_{ij} \) apply whenever state \( i \) is visited, no matter what happened in the past, and no matter how state \( i \) was reached. Mathematically, we assume the Markov property, which requires that

\[
\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \Pr(X_{n+1} = j \mid X_n = i) = p_{ij},
\]

for all times \( n \), all states \( i, j \in S \) and all possible sequences \( i_0, \ldots, i_{n-1} \) of earlier states. Thus, the probability law of the next state \( X_{n+1} \) depends on the past only through the value of the present state \( X_n \).

The transition probabilities \( p_{ij} \) must be of course nonnegative, and sum to one:

\[
\sum_{j=1}^{m} p_{ij} = 1, \quad \text{for all } i.
\]
We will generally allow the probabilities $p_{ii}$ to be positive, in which case it is possible for the next state to be the same as the current one. Even though the state does not change, we still view this as a state transition of a special type (a "self-transition").

**Specification of Markov Models**

- A Markov chain model is specified by identifying:
  - (a) the set of states $S = \{1, \ldots, m\}$,
  - (b) the set of possible transitions, namely, those pairs $(i, j)$ for which $p_{ij} > 0$, and,
  - (c) the numerical values of those $p_{ij}$ that are positive.
- The Markov chain specified by this model is a sequence of random variables $X_0, X_1, X_2, \ldots$, that take values in $S$, and which satisfy
  $$
P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = p_{ij},$$
  for all times $n$, all states $i, j \in S$, and all possible sequences $i_0, \ldots, i_{n-1}$ of earlier states.

All of the elements of a Markov chain model can be encoded in a **transition probability matrix**, which is simply a two-dimensional array whose element at the $i$th row and $j$th column is $p_{ij}$:

$$
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1m} \\
p_{21} & p_{22} & \cdots & p_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{m1} & p_{m2} & \cdots & p_{mm}
\end{bmatrix}$$

It is also helpful to lay out the model in the so-called **transition probability graph**, whose nodes are the states and whose arcs are the possible transitions. By recording the numerical values of $p_{ij}$ near the corresponding arcs, one can visualize the entire model in a way that can make some of its major properties readily apparent.

**Example 7.1.** Alice is taking a probability class and in each week, she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in the given week, the probability that she will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). We assume that these probabilities do not depend on whether she was up-to-date or behind in previous weeks, so the problem has the typical Markov chain character (the future depends on the past only through the present).
Let us introduce states 1 and 2, and identify them with being up-to-date and behind, respectively. Then, the transition probabilities are

\[ p_{11} = 0.8, \quad p_{12} = 0.2, \quad p_{21} = 0.6, \quad p_{22} = 0.4, \]

and the transition probability matrix is

\[
\begin{bmatrix}
0.8 & 0.2 \\
0.6 & 0.4
\end{bmatrix}.
\]

The transition probability graph is shown in Fig. 7.1.

![Transition probability graph](image)

**Figure 7.1:** The transition probability graph in Example 7.1.

**Example 7.2. Spiders and Fly.** A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independent of the past history of movements. Two spiders are lurking at positions 1 and \( m \): if the fly lands there, it is captured by a spider, and the process terminates. We want to construct a Markov chain model, assuming that the fly starts in a position between 1 and \( m \).

Let us introduce states 1, 2, \ldots, \( m \), and identify them with the corresponding positions of the fly. The nonzero transition probabilities are

\[ p_{11} = 1, \quad p_{m m} = 1, \]

\[ p_{ij} = \begin{cases} 
0.3, & \text{if } j = i - 1 \text{ or } j = i + 1, \\
0.4, & \text{if } j = i,
\end{cases} \quad \text{for } i = 2, \ldots, m - 1.\]

The transition probability graph and matrix are shown in Fig. 7.2.

![Transition probability graph](image)

**Figure 7.2:** The transition probability graph and the transition probability matrix in Example 7.2, for the case where \( m = 4 \).
Example 7.3. Machine Failure, Repair, and Replacement. A machine can be either working or broken down on a given day. If it is working, it will break down in the next day with probability $b$, and will continue working with probability $1 - b$. If it breaks down on a given day, it will be repaired and be working in the next day with probability $r$, and will continue to be broken down with probability $1 - r$.

We model the machine by a Markov chain with the following two states:

State 1: Machine is working, State 2: Machine is broken down.

The transition probability graph of the chain is given in Fig. 7.3. The transition probability matrix is

$$
\begin{bmatrix}
1 - b & b \\
r & 1 - r
\end{bmatrix}.
$$

![Figure 7.3: Transition probability graph for Example 7.3.](image)

The situation considered here evidently has the Markov property: the state of the machine at the next day depends explicitly only on its state at the present day. However, it is possible to use a Markov chain model even if there is a dependence on the states at several past days. The general idea is to introduce some additional states which encode relevant information from preceding periods, as in the variation that we consider next.

Suppose that whenever the machine remains broken for a given number of $\ell$ days, despite the repair efforts, it is replaced by a new working machine. To model this as a Markov chain, we replace the single state 2, corresponding to a broken down machine, with several states that indicate the number of days that the machine is broken. These states are

State $(2,i)$: The machine has been broken for $i$ days, $i = 1, 2, \ldots, \ell$.

The transition probability graph is given in Fig. 7.4 for the case where $\ell = 4$.

The second half of the preceding example illustrates that in order to construct a Markov model, there is often a need to introduce new states that capture the dependence of the future on the model's past history. We note that there is some freedom in selecting these additional states, but their number should be generally kept small, for reasons of analytical or computational tractability.

The Probability of a Path

Given a Markov chain model, we can compute the probability of any particular sequence of future states. This is analogous to the use of the multiplication rule
in sequential (tree) probability models. In particular, we have

\[ P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_0)p_{i_0i_1}p_{i_1i_2} \cdots p_{i_{n-1}i_n}. \]

To verify this property, note that

\[
P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_n = i_n \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1})P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1})
\]

\[ = p_{i_{n-1}i_n}P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}), \]

where the last equality made use of the Markov property. We then apply the same argument to the term \(P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1})\) and continue similarly, until we eventually obtain the desired expression. If the initial state \(X_0\) is given and is known to be equal to some \(i_0\), a similar argument yields

\[ P(X_1 = i_1, \ldots, X_n = i_n \mid X_0 = i_0) = p_{i_0i_1}p_{i_1i_2} \cdots p_{i_{n-1}i_n}. \]

Graphically, a state sequence can be identified with a sequence of arcs in the transition probability graph, and the probability of such a path (given the initial state) is given by the product of the probabilities associated with the arcs traversed by the path.

**Example 7.4.** For the spider and fly example (Example 7.2), we have

\[ P(X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4 \mid X_0 = 2) = p_{22}p_{22}p_{23}p_{34} = (0.4)^2(0.3)^2. \]

We also have

\[
P(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = P(X_0 = 2)p_{22}p_{22}p_{23}p_{34}
\]

\[ = P(X_0 = 2)(0.4)^2(0.3)^2. \]

Note that in order to calculate a probability of this form, in which there is no conditioning on a fixed initial state, we need to specify a probability law for the initial state \(X_0\).
n-Step Transition Probabilities

Many Markov chain problems require the calculation of the probability law of the state at some future time, conditioned on the current state. This probability law is captured by the \textit{n-step transition probabilities}, defined by

\[ r_{ij}(n) = P(X_n = j \mid X_0 = i). \]

In words, \( r_{ij}(n) \) is the probability that the state after \( n \) time periods will be \( j \), given that the current state is \( i \). It can be calculated using the following basic recursion, known as the \textbf{Chapman-Kolmogorov equation}.

\begin{equation}
\text{Chapman-Kolmogorov Equation for the n-Step Transition Probabilities}
\end{equation}

The \( n \)-step transition probabilities can be generated by the recursive formula

\[ r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n - 1)p_{kj}, \quad \text{for } n > 1, \text{ and all } i, j, \]

starting with

\[ r_{ij}(1) = p_{ij}. \]

To verify the formula, we apply the total probability theorem as follows:

\[ P(X_n = j \mid X_0 = i) = \sum_{k=1}^{m} P(X_{n-1} = k \mid X_0 = i) P(X_n = j \mid X_{n-1} = k, X_0 = i) \]

\[ = \sum_{k=1}^{m} r_{ik}(n - 1)p_{kj}; \]

see Fig. 7.5 for an illustration. We have used here the Markov property: once we condition on \( X_{n-1} = k \), the conditioning on \( X_0 = i \) does not affect the probability \( p_{kj} \) of reaching \( j \) at the next step.

We can view \( r_{ij}(n) \) as the element at the \( i \)th row and \( j \)th column of a two-dimensional array, called the \textbf{n-step transition probability matrix}.† Figures 7.6 and 7.7 give the \( n \)-step transition probabilities \( r_{ij}(n) \) for the cases of

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† Those readers familiar with matrix multiplication, may recognize that the Chapman-Kolmogorov equation can be expressed as follows: the matrix of \( n \)-step transition probabilities \( r_{ij}(n) \) is obtained by multiplying the matrix of \((n - 1)\)-step transition probabilities \( r_{ik}(n - 1) \), with the one-step transition probability matrix. Thus, the \( n \)-step transition probability matrix is the \( n \)th power of the transition probability matrix.
Examples 7.1 and 7.2, respectively. There are some interesting observations about the limiting behavior of \( r_{ij}(n) \) in these two examples. In Fig. 7.6, we see that each \( r_{ij}(n) \) converges to a limit, as \( n \to \infty \), and this limit does not depend on the initial state \( i \). Thus, each state has a positive “steady-state” probability of being occupied at times far into the future. Furthermore, the probability \( r_{ij}(n) \) depends on the initial state \( i \) when \( n \) is small, but over time this dependence diminishes. Many (but by no means all) probabilistic models that evolve over time have such a character: after a sufficiently long time, the effect of their initial condition becomes negligible.

In Fig. 7.7, we see a qualitatively different behavior: \( r_{ij}(n) \) again converges, but the limit depends on the initial state, and can be zero for selected states. Here, we have two states that are “absorbing,” in the sense that they are infinitely repeated, once reached. These are the states 1 and 4 that correspond to the capture of the fly by one of the two spiders. Given enough time, it is certain that some absorbing state will be reached. Accordingly, the probability of being at the non-absorbing states 2 and 3 diminishes to zero as time increases. Furthermore, the probability that a particular absorbing state will be reached depends on how “close” we start to that state.

These examples illustrate that there is a variety of types of states and asymptotic occupancy behavior in Markov chains. We are thus motivated to classify and analyze the various possibilities, and this is the subject of the next three sections.

### 7.2 CLASSIFICATION OF STATES

In the preceding section, we saw some examples indicating that the various states of a Markov chain can have qualitatively different characteristics. In particular,
some states, after being visited once, are certain to be visited again, while for some other states this may not be the case. In this section, we focus on the mechanism by which this occurs. In particular, we wish to classify the states of a Markov chain with a focus on the long-term frequency with which they are visited.

As a first step, we make the notion of revisiting a state precise. Let us say that a state $j$ is accessible from a state $i$ if for some $n$, the $n$-step transition probability $r_{ij}(n)$ is positive, i.e., if there is positive probability of reaching $j$, starting from $i$, after some number of time periods. An equivalent definition is that there is a possible state sequence $i, i_1, \ldots, i_{n-1}, j$, that starts at $i$ and ends at $j$, in which the transitions $(i, i_1), (i_1, i_2), \ldots, (i_{n-2}, i_{n-1}), (i_{n-1}, j)$ all have positive probability. Let $A(i)$ be the set of states that are accessible from $i$. We say that $i$ is recurrent if for every $j$ that is accessible from $i$, $i$ is also accessible from $j$; that is, for all $j$ that belong to $A(i)$ we have that $i$ belongs to $A(j)$.

When we start at a recurrent state $i$, we can only visit states $j \in A(i)$ from which $i$ is accessible. Thus, from any future state, there is always some probability of returning to $i$ and, given enough time, this is certain to happen. By repeating this argument, if a recurrent state is visited once, it is certain to be revisited an infinite number of times. (See the end-of-chapter problems for a more rigorous version of this argument.)

A state is called transient if it is not recurrent. Thus, a state $i$ is transient
Figure 7.7: The top part of the figure shows the $n$-step transition probabilities $r_{11}(n)$ for the "spiders-and-fly" Example 7.2. These are the probabilities of reaching state 1 by time $n$, starting from state $i$. We observe that these probabilities converge to a limit, but the limit depends on the starting state. In this example, note that the probabilities $r_{23}(n)$ and $r_{33}(n)$ of being at the non-absorbing states 2 or 3, go to zero, as $n$ increases.

if there is a state $j \in A(i)$ such that $i$ is not accessible from $j$. After each visit to state $i$, there is positive probability that the state enters such a $j$. Given enough time, this will happen, and state $i$ cannot be visited after that. Thus, a transient state will only be visited a finite number of times; see again the end-of-chapter problems.

Note that transience or recurrence is determined by the arcs of the transition probability graph [those pairs $(i, j)$ for which $p_{ij} > 0$] and not by the numerical values of the $p_{ij}$. Figure 7.8 provides an example of a transition probability graph, and the corresponding recurrent and transient states.

If $i$ is a recurrent state, the set of states $A(i)$ that are accessible from $i$ form a **recurrent class** (or simply **class**), meaning that states in $A(i)$ are all accessible from each other, and no state outside $A(i)$ is accessible from them. Mathematically, for a recurrent state $i$, we have $A(i) = A(j)$ for all $j$ that belong to $A(i)$, as can be seen from the definition of recurrence. For example, in the graph of Fig. 7.8, states 3 and 4 form a class, and state 1 by itself also forms a class.

It can be seen that at least one recurrent state must be accessible from any
Figure 7.8: Classification of states given the transition probability graph. Starting from state 1, the only accessible state is itself, and so 1 is a recurrent state. States 1, 3, and 4 are accessible from 2, but 2 is not accessible from any of them, so state 2 is transient. States 3 and 4 are accessible from each other, and they are both recurrent.

given transient state. This is intuitively evident and is left as an end-of-chapter problem. It follows that there must exist at least one recurrent state, and hence at least one class. Thus, we reach the following conclusion.

Markov Chain Decomposition

- A Markov chain can be decomposed into one or more recurrent classes, plus possibly some transient states.
- A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
- A transient state is not accessible from any recurrent state.
- At least one, possibly more, recurrent states are accessible from a given transient state.

Figure 7.9 provides examples of Markov chain decompositions. Decomposition provides a powerful conceptual tool for reasoning about Markov chains and visualizing the evolution of their state. In particular, we see that:

(a) once the state enters (or starts in) a class of recurrent states, it stays within that class; since all states in the class are accessible from each other, all states in the class will be visited an infinite number of times:

(b) if the initial state is transient, then the state trajectory contains an initial portion consisting of transient states and a final portion consisting of recurrent states from the same class.

For the purpose of understanding long-term behavior of Markov chains, it is important to analyze chains that consist of a single recurrent class. For the purpose of understanding short-term behavior, it is also important to analyze the mechanism by which any particular class of recurrent states is entered starting from a given transient state. These two issues, long-term and short-term behavior, are the focus of Sections 7.3 and 7.4, respectively.
Periodicity

There is another important characterization of a recurrent class, which relates to the presence or absence of a certain periodic pattern in the times that a state can be visited. In particular, a recurrent class is said to be **periodic** if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$ so that all transitions from one subset lead to the next subset: see Fig. 7.10. More precisely,

$$
\text{if } i \in S_k \text{ and } p_{ij} > 0. \quad \text{then } \begin{cases} 
  j \in S_{k+1} & \text{if } k = 1, \ldots, d - 1, \\
  j \in S_1 & \text{if } k = d.
\end{cases}
$$

A recurrent class that is not periodic is said to be **aperiodic.**
Figure 7.10: Structure of a periodic recurrent class. In this example, $d = 3$.

Thus, in a periodic recurrent class, we move through the sequence of subsets in order, and after $d$ steps, we end up in the same subset. As an example, the recurrent class in the second chain of Fig. 7.9 (states 1 and 2) is periodic, and the same is true of the class consisting of states 4 and 5 in the third chain of Fig. 7.9. All other recurrent classes in the chains of this figure are aperiodic.

Note that given a periodic recurrent class, a positive time $n$, and a state $i$ in the class, there must exist one or more states $j$ for which $r_{ij}(n) = 0$. The reason is that starting from $i$, only one of the sets $S_k$ is possible at time $n$. Thus, a way to verify aperiodicity of a given recurrent class $R$, is to check whether there is a special time $n \geq 1$ and a special state $i \in R$ from which all states in $R$ can be reached in $n$ steps, i.e., $r_{ij}(n) > 0$ for all $j \in R$. As an example, consider the first chain in Fig. 7.9. Starting from state 1, every state is possible at time $n = 3$, so the unique recurrent class of that chain is aperiodic.

A converse statement, which we do not prove, also turns out to be true: if a recurrent class $R$ is aperiodic, then there exists a time $n$ such that $r_{ij}(n) > 0$ for every $i$ and $j$ in $R$, see the end-of-chapter problems.

### Periodicity

Consider a recurrent class $R$.

- The class is called **periodic** if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$, so that all transitions from $S_k$ lead to $S_{k+1}$ (or to $S_1$ if $k = d$).

- The class is **aperiodic** (not periodic) if and only if there exists a time $n$ such that $r_{ij}(n) > 0$, for all $i, j \in R$. 

7.3 STEADY-STATE BEHAVIOR

In Markov chain models, we are often interested in long-term state occupancy behavior, that is, in the $n$-step transition probabilities $r_{ij}(n)$ when $n$ is very large. We have seen in the example of Fig. 7.6 that the $r_{ij}(n)$ may converge to steady-state values that are independent of the initial state. We wish to understand the extent to which this behavior is typical.

If there are two or more classes of recurrent states, it is clear that the limiting values of the $r_{ij}(n)$ must depend on the initial state (the possibility of visiting $j$ far into the future depends on whether $j$ is in the same class as the initial state $i$). We will, therefore, restrict attention to chains involving a single recurrent class, plus possibly some transient states. This is not as restrictive as it may seem, since we know that once the state enters a particular recurrent class, it will stay within that class. Thus, the asymptotic behavior of a multiclass chain can be understood in terms of the asymptotic behavior of a single-class chain.

Even for chains with a single recurrent class, the $r_{ij}(n)$ may fail to converge. To see this, consider a recurrent class with two states, 1 and 2, such that from state 1 we can only go to 2, and from 2 we can only go to 1 ($p_{12} = p_{21} = 1$). Then, starting at some state, we will be in that same state after any even number of transitions, and in the other state after any odd number of transitions. Formally,

$$r_{ii}(n) = \begin{cases} 1, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

What is happening here is that the recurrent class is periodic, and for such a class, it can be seen that the $r_{ij}(n)$ generically oscillate.

We now assert that for every state $j$, the probability $r_{ij}(n)$ of being at state $j$ approaches a limiting value that is independent of the initial state $i$, provided we exclude the two situations discussed above (multiple recurrent classes and/or a periodic class). This limiting value, denoted by $\pi_j$, has the interpretation

$$\pi_j \approx P(X_n = j), \quad \text{when } n \text{ is large,}$$

and is called the steady-state probability of $j$. The following is an important theorem. Its proof is quite complicated and is outlined together with several other proofs in the end-of-chapter problems.

<table>
<thead>
<tr>
<th>Steady-State Convergence Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider a Markov chain with a single recurrent class, which is aperiodic. Then, the states $j$ are associated with steady-state probabilities $\pi_j$ that have the following properties.</td>
</tr>
</tbody>
</table>
Sec. 7.3  Steady-State Behavior

(a) For each \( j \), we have

\[
\lim_{n \to \infty} r_{ij}(n) = \pi_j, \quad \text{for all } i.
\]

(b) The \( \pi_j \) are the unique solution to the system of equations below:

\[
\begin{align*}
\pi_j &= \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, \ldots, m, \\
1 &= \sum_{k=1}^{m} \pi_k.
\end{align*}
\]

(c) We have

\[
\begin{align*}
\pi_j &= 0, \quad \text{for all transient states } j, \\
\pi_j &= 0, \quad \text{for all recurrent states } j.
\end{align*}
\]

The steady-state probabilities \( \pi_j \) sum to 1 and form a probability distribution on the state space, called the **stationary distribution** of the chain. The reason for the qualification "stationary" is that if the initial state is chosen according to this distribution, i.e., if

\[
P(X_0 = j) = \pi_j, \quad j = 1, \ldots, m,
\]

then, using the total probability theorem, we have

\[
P(X_1 = j) = \sum_{k=1}^{m} P(X_0 = k)p_{kj} = \sum_{k=1}^{m} \pi_k p_{kj} = \pi_j,
\]

where the last equality follows from part (b) of the steady-state convergence theorem. Similarly, we obtain \( P(X_n = j) = \pi_j \), for all \( n \) and \( j \). Thus, if the initial state is chosen according to the stationary distribution, the state at any future time will have the same distribution.

The equations

\[
\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, \ldots, m,
\]

are called the **balance equations**. They are a simple consequence of part (a) of the theorem and the Chapman-Kolmogorov equation. Indeed, once the convergence of \( r_{ij}(n) \) to some \( \pi_j \) is taken for granted, we can consider the equation,

\[
r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n - 1)p_{kj},
\]
take the limit of both sides as \( n \to \infty \), and recover the balance equations.† Together with the **normalization equation**

\[
\sum_{k=1}^{m} \pi_k = 1,
\]

the balance equations can be solved to obtain the \( \pi_j \). The following examples illustrate the solution process.

**Example 7.5.** Consider a two-state Markov chain with transition probabilities

\[
\begin{align*}
p_{11} &= 0.8, & p_{12} &= 0.2, \\
p_{21} &= 0.6, & p_{22} &= 0.4.
\end{align*}
\]

(This is the same as the chain of Example 7.1 and Fig. 7.1.) The balance equations take the form

\[
\begin{align*}
\pi_1 &= \pi_1 p_{11} + \pi_2 p_{21}, \\
\pi_2 &= \pi_1 p_{12} + \pi_2 p_{22},
\end{align*}
\]

or

\[
\begin{align*}
\pi_1 &= 0.8 \cdot \pi_1 + 0.6 \cdot \pi_2, \\
\pi_2 &= 0.2 \cdot \pi_1 + 0.4 \cdot \pi_2.
\end{align*}
\]

Note that the above two equations are dependent, since they are both equivalent to

\[\pi_1 = 3\pi_2.\]

This is a generic property, and in fact it can be shown that any one of the balance equations can always be derived from the remaining equations. However, we also know that the \( \pi_j \) satisfy the normalization equation

\[\pi_1 + \pi_2 = 1,\]

which supplements the balance equations and suffices to determine the \( \pi_j \) uniquely. Indeed, by substituting the equation \( \pi_1 = 3\pi_2 \) into the equation \( \pi_1 + \pi_2 = 1 \), we obtain \( 3\pi_2 + \pi_2 = 1 \), or

\[\pi_2 = 0.25,\]

which using the equation \( \pi_1 + \pi_2 = 1 \), yields

\[\pi_1 = 0.75.\]

This is consistent with what we found earlier by iterating the Chapman-Kolmogorov equation (cf. Fig. 7.6).

† According to a famous and important theorem from linear algebra (called the Perron-Frobenius theorem), the balance equations always have a nonnegative solution, for any Markov chain. What is special about a chain that has a single recurrent class, which is aperiodic, is that given also the normalization equation, the solution is unique and is equal to the limit of the \( n \)-step transition probabilities \( r_{ij}(n) \).
Example 7.6. An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability \( p \) each time she commutes, independent of other times. What is the steady-state probability that she gets wet during a commute?

We model this problem using a Markov chain with the following states:

State \( i \): \( i \) umbrellas are available in her current location, \( i = 0, 1, 2 \).

The transition probability graph is given in Fig. 7.11, and the transition probability matrix is

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 - p & p \\
1 - p & p & 0
\end{bmatrix}.
\]

![Transition probability graph for Example 7.6.](image)

The chain has a single recurrent class that is aperiodic (assuming \( 0 < p < 1 \)), so the steady-state convergence theorem applies. The balance equations are

\[
\pi_0 = (1 - p)\pi_2, \quad \pi_1 = (1 - p)\pi_1 + p\pi_2, \quad \pi_2 = \pi_0 + p\pi_1.
\]

From the second equation, we obtain \( \pi_1 = \pi_2 \), which together with the first equation \( \pi_0 = (1 - p)\pi_2 \) and the normalization equation \( \pi_0 + \pi_1 + \pi_2 = 1 \), yields

\[
\pi_0 = \frac{1 - p}{3 - p}, \quad \pi_1 = \frac{1}{3 - p}, \quad \pi_2 = \frac{1}{3 - p}.
\]

According to the steady-state convergence theorem, the steady-state probability that the professor finds herself in a place without an umbrella is \( \pi_0 \). The steady-state probability that she gets wet is \( \pi_0 \) times the probability of rain \( p \).

Example 7.7. A superstitious professor works in a circular building with \( m \) doors, where \( m \) is odd, and never uses the same door twice in a row. Instead he uses with probability \( p \) (or probability \( 1 - p \)) the door that is adjacent in the clockwise direction (or the counterclockwise direction, respectively) to the door he used last.
What is the probability that a given door will be used on some particular day far into the future? We introduce a Markov chain with the following $m$ states:

State $i$: last door used is door $i$, $i = 1, \ldots, m$.

The transition probability graph of the chain is given in Fig. 7.12, for the case $m = 5$. The transition probability matrix is

$$
\begin{bmatrix}
0 & p & 0 & 0 & \ldots & 0 & 1 - p \\
1 - p & 0 & p & 0 & \ldots & 0 & 0 \\
0 & 1 - p & 0 & p & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p & 0 & 0 & 0 & \ldots & 1 - p & 0
\end{bmatrix}
$$

Assuming that $0 < p < 1$, the chain has a single recurrent class that is aperiodic. (To verify aperiodicity, we leave it to the reader to verify that given an initial state, every state $j$ can be reached in exactly $m$ steps, and the criterion for aperiodicity given at the end of the preceding section is satisfied.) The balance equations are

$$
\pi_1 = (1 - p)\pi_2 + p\pi_m, \\
\pi_i = p\pi_{i-1} + (1 - p)\pi_{i+1}, \quad i = 2, \ldots, m - 1, \\
\pi_m = (1 - p)\pi_1 + p\pi_{m-1}.
$$

These equations are easily solved once we observe that by symmetry, all doors
should have the same steady-state probability. This suggests the solution

\[ \pi_j = \frac{1}{m}, \quad j = 1, \ldots, m. \]

Indeed, we see that these \( \pi_j \) satisfy the balance equations as well as the normalization equation, so they must be the desired steady-state probabilities (by the uniqueness part of the steady-state convergence theorem).

Note that if either \( p = 0 \) or \( p = 1 \), the chain still has a single recurrent class but is periodic. In this case, the \( n \)-step transition probabilities \( \tau_{ij}(n) \) do not converge to a limit, because the doors are used in a cyclic order. Similarly, if \( m \) is even, the recurrent class of the chain is periodic, since the states can be grouped into two subsets, the even and the odd numbered states, such that from each subset one can only go to the other subset.

**Long-Term Frequency Interpretations**

Probabilities are often interpreted as relative frequencies in an infinitely long string of independent trials. The steady-state probabilities of a Markov chain admit a similar interpretation, despite the absence of independence.

Consider, for example, a Markov chain involving a machine, which at the end of any day can be in one of two states, working or broken down. Each time it breaks down, it is immediately repaired at a cost of \$1. How are we to model the long-term expected cost of repair per day? One possibility is to view it as the expected value of the repair cost on a randomly chosen day far into the future; this is just the steady-state probability of the broken down state. Alternatively, we can calculate the total expected repair cost in \( n \) days, where \( n \) is very large, and divide it by \( n \). Intuition suggests that these two methods of calculation should give the same result. Theory supports this intuition, and in general we have the following interpretation of steady-state probabilities (a justification is given in the end-of-chapter problems).

**Steady-State Probabilities as Expected State Frequencies**

For a Markov chain with a single class which is aperiodic, the steady-state probabilities \( \pi_j \) satisfy

\[ \pi_j = \lim_{n \to \infty} \frac{v_{ij}(n)}{n}, \]

where \( v_{ij}(n) \) is the expected value of the number of visits to state \( j \) within the first \( n \) transitions, starting from state \( i \).

Based on this interpretation, \( \pi_j \) is the long-term expected fraction of time that the state is equal to \( j \). Each time that state \( j \) is visited, there is probability \( p_{jk} \) that the next transition takes us to state \( k \). We conclude that \( \pi_j p_{jk} \) can
be viewed as the long-term expected fraction of transitions that move the state from $j$ to $k$.\footnote{In fact, some stronger statements are also true, such as the following. Whenever we carry out a probabilistic experiment and generate a trajectory of the Markov chain over an infinite time horizon, the observed long-term frequency with which state $j$ is visited will be exactly equal to $\pi_j$, and the observed long-term frequency of transitions from $j$ to $k$ will be exactly equal to $\pi_jp_{jk}$. Even though the trajectory is random, these equalities hold with essential certainty, that is, with probability 1.}

\begin{center}
\textbf{Expected Frequency of a Particular Transition}
\end{center}

Consider $n$ transitions of a Markov chain with a single class which is aperiodic, starting from a given initial state. Let $q_{jk}(n)$ be the expected number of such transitions that take the state from $j$ to $k$. Then, regardless of the initial state, we have

$$\lim_{n \to \infty} \frac{q_{jk}(n)}{n} = \pi_jp_{jk}.$$ 

Given the frequency interpretation of $\pi_j$ and $\pi_kp_{kj}$, the balance equation

$$\pi_j = \sum_{k=1}^{m} \pi_kp_{kj}$$

has an intuitive meaning. It expresses the fact that the expected frequency $\pi_j$ of visits to $j$ is equal to the sum of the expected frequencies $\pi_kp_{kj}$ of transitions that lead to $j$; see Fig. 7.13.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure713.png}
\caption{Interpretation of the balance equations in terms of frequencies. In a very large number of transitions, we expect a fraction $\pi_kp_{kj}$ that bring the state from $k$ to $j$. (This also applies to transitions from $j$ to itself, which occur with frequency $\pi_jp_{jj}$.) The sum of the expected frequencies of such transitions is the expected frequency $\pi_j$ of being at state $j$.}
\end{figure}

Chap. 7

Markov Chains
Birth-Death Processes

A birth-death process is a Markov chain in which the states are linearly arranged and transitions can only occur to a neighboring state, or else leave the state unchanged. They arise in many contexts, especially in queueing theory. Figure 7.14 shows the general structure of a birth-death process and also introduces some generic notation for the transition probabilities. In particular,

\[ b_i = P(X_{n+1} = i + 1 | X_n = i), \quad (\text{"birth" probability at state } i), \]
\[ d_i = P(X_{n+1} = i - 1 | X_n = i), \quad (\text{"death" probability at state } i). \]

![Transition probability graph for a birth-death process.](image)

Figure 7.14: Transition probability graph for a birth-death process.

For a birth-death process, the balance equations can be substantially simplified. Let us focus on two neighboring states, say, \( i \) and \( i + 1 \). In any trajectory of the Markov chain, a transition from \( i \) to \( i + 1 \) has to be followed by a transition from \( i + 1 \) to \( i \), before another transition from \( i \) to \( i + 1 \) can occur. Therefore, the expected frequency of transitions from \( i \) to \( i + 1 \), which is \( \pi_i b_i \), must be equal to the expected frequency of transitions from \( i + 1 \) to \( i \), which is \( \pi_{i+1} d_{i+1} \). This leads to the local balance equations\(^\dagger\)

\[ \pi_i b_i = \pi_{i+1} d_{i+1}, \quad i = 0, 1, \ldots, m - 1. \]

Using the local balance equations, we obtain

\[ \pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}, \quad i = 1, \ldots, m. \]

from which, using also the normalization equation \( \sum_i \pi_i = 1 \), the steady-state probabilities \( \pi_i \) are easily computed.

\(^\dagger\) A more formal derivation that does not rely on the frequency interpretation proceeds as follows. The balance equation at state 0 is \( \pi_0 (1 - b_0) + \pi_1 d_1 = \pi_0 \), which yields the first local balance equation \( \pi_0 b_0 = \pi_1 d_1 \).

The balance equation at state 1 is \( \pi_0 b_0 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 = \pi_1 \). Using the local balance equation \( \pi_0 b_0 = \pi_1 d_1 \) at the previous state, this is rewritten as \( \pi_1 d_1 + \pi_1 (1 - b_1 - d_1) + \pi_2 d_2 = \pi_1 \), which simplifies to \( \pi_1 b_1 = \pi_2 d_2 \). We can then continue similarly to obtain the local balance equations at all other states.
**Example 7.8. Random Walk with Reflecting Barriers.** A person walks along a straight line and, at each time period, takes a step to the right with probability \( b \) and a step to the left with probability \( 1 - b \). The person starts in one of the positions \( 1, 2, \ldots, m \), but if he reaches position 0 (or position \( m + 1 \)), his step is instantly reflected back to position 1 (or position \( m \), respectively). Equivalently, we may assume that when the person is in positions 1 or \( m \), he will stay in that position with corresponding probability \( 1 - b \) and \( b \), respectively. We introduce a Markov chain model whose states are the positions \( 1, \ldots, m \). The transition probability graph of the chain is given in Fig. 7.15.

![Transition probability graph](image)

**Figure 7.15:** Transition probability graph for the random walk Example 7.8.

The local balance equations are

\[ \pi_i b = \pi_{i+1} (1 - b), \quad i = 1, \ldots, m - 1. \]

Thus, \( \pi_{i+1} = \rho \pi_i \), where

\[ \rho = \frac{b}{1 - b}, \]

and we can express all the \( \pi_j \) in terms of \( \pi_1 \), as

\[ \pi_i = \rho^{i-1} \pi_1, \quad i = 1, \ldots, m. \]

Using the normalization equation \( 1 = \pi_1 + \cdots + \pi_m \), we obtain

\[ 1 = \pi_1 (1 + \rho + \cdots + \rho^{m-1}) \]

which leads to

\[ \pi_i = \frac{\rho^{i-1}}{1 + \rho + \cdots + \rho^{m-1}}, \quad i = 1, \ldots, m. \]

Note that if \( \rho = 1 \) (left and right steps are equally likely), then \( \pi_i = 1/m \) for all \( i \).

**Example 7.9. Queueing.** Packets arrive at a node of a communication network, where they are stored in a buffer and then transmitted. The storage capacity of the buffer is \( m \): if \( m \) packets are already present, any newly arriving packets are discarded. We discretize time in very small periods, and we assume that in each period, at most one event can happen that can change the number of packets stored in the node (an arrival of a new packet or a completion of the transmission of an
existing packet). In particular, we assume that at each period, exactly one of the following occurs:

(a) one new packet arrives; this happens with a given probability $b > 0$;

(b) one existing packet completes transmission; this happens with a given probability $d > 0$ if there is at least one packet in the node, and with probability 0 otherwise;

(c) no new packet arrives and no existing packet completes transmission; this happens with probability $1 - b - d$ if there is at least one packet in the node, and with probability $1 - b$ otherwise.

We introduce a Markov chain with states $0, 1, \ldots, m$, corresponding to the number of packets in the buffer. The transition probability graph is given in Fig. 7.16.

![Transition probability graph in Example 7.9.](image)

The local balance equations are

$$ \pi_i b = \pi_{i+1} d, \quad i = 0, 1, \ldots, m - 1. $$

We define

$$ \rho = \frac{b}{d}. $$

and obtain $\pi_{i+1} = \rho \pi_i$, which leads to

$$ \pi_i = \rho^i \pi_0, \quad i = 0, 1, \ldots, m. $$

By using the normalization equation $1 = \pi_0 + \pi_1 + \cdots + \pi_m$, we obtain

$$ 1 = \pi_0(1 + \rho + \cdots + \rho^m), $$

and

$$ \pi_0 = \begin{cases} 
\frac{1 - \rho}{1 - \rho^{m+1}}, & \text{if } \rho \neq 1, \\
\frac{1}{m+1}, & \text{if } \rho = 1.
\end{cases} $$

Using the equation $\pi_i = \rho^i \pi_0$, the steady-state probabilities are

$$ \pi_i = \begin{cases} 
\frac{1 - \rho}{1 - \rho^{m+1}} \rho^i, & \text{if } \rho \neq 1, \\
\frac{1}{m+1}, & \text{if } \rho = 1,
\end{cases} \quad i = 0, 1, \ldots, m. $$
It is interesting to consider what happens when the buffer size $m$ is so large that it can be considered as practically infinite. We distinguish two cases.

(a) Suppose that $b < d$, or $\rho < 1$. In this case, arrivals of new packets are less likely than departures of existing packets. This prevents the number of packets in the buffer from growing, and the steady-state probabilities $\pi_i$ decrease with $i$, as in a (truncated) geometric PMF. We observe that as $m \to \infty$, we have $1 - \rho^{m+1} \to 1$, and

$$
\pi_i \to \rho^i (1 - \rho), \quad \text{for all } i.
$$

We can view these as the steady-state probabilities in a system with an infinite buffer. [As a check, note that we have $\sum_{i=0}^{\infty} \rho^i (1 - \rho) = 1$.]

(b) Suppose that $b > d$, or $\rho > 1$. In this case, arrivals of new packets are more likely than departures of existing packets. The number of packets in the buffer tends to increase, and the steady-state probabilities $\pi_i$ increase with $i$. As we consider larger and larger buffer sizes $m$, the steady-state probability of any fixed state $i$ decreases to zero:

$$
\pi_i \to 0, \quad \text{for all } i.
$$

Were we to consider a system with an infinite buffer, we would have a Markov chain with a countably infinite number of states. Although we do not have the machinery to study such chains, the preceding calculation suggests that every state will have zero steady-state probability and will be "transient." The number of packets in queue will generally grow to infinity, and any particular state will be visited only a finite number of times.

The preceding analysis provides a glimpse into the character of Markov chains with an infinite number of states. In such chains, even if there is a single and aperiodic recurrent class, the chain may never reach steady-state and a steady-state distribution may not exist.

### 7.4 Absorption Probabilities and Expected Time to Absorption

In this section, we study the short-term behavior of Markov chains. We first consider the case where the Markov chain starts at a transient state. We are interested in the first recurrent state to be entered, as well as in the time until this happens.

When addressing such questions, the subsequent behavior of the Markov chain (after a recurrent state is encountered) is immaterial. We can therefore focus on the case where every recurrent state $k$ is absorbing, i.e.,

$$
p_{kk} = 1, \quad p_{kj} = 0 \quad \text{for all } j \neq k.
$$

If there is a unique absorbing state $k$, its steady-state probability is 1 (because all other states are transient and have zero steady-state probability), and will be
reached with probability 1, starting from any initial state. If there are multiple absorbing states, the probability that one of them will be eventually reached is still 1, but the identity of the absorbing state to be entered is random and the associated probabilities may depend on the starting state. In the sequel, we fix a particular absorbing state, denoted by $s$, and consider the absorption probability $a_i$ that $s$ is eventually reached, starting from $i$:

$$a_i = P(X_n \text{ eventually becomes equal to the absorbing state } s \mid X_0 = i).$$

Absorption probabilities can be obtained by solving a system of linear equations, as indicated below.

### Absorption Probability Equations

Consider a Markov chain where each state is either transient or absorbing, and fix a particular absorbing state $s$. Then, the probabilities $a_i$ of eventually reaching state $s$, starting from $i$, are the unique solution to the equations

$$a_s = 1,$$

$$a_i = 0, \quad \text{for all absorbing } i \neq s,$$

$$a_i = \sum_{j=1}^{m} p_{ij}a_j, \quad \text{for all transient } i.$$

The equations $a_s = 1$, and $a_i = 0$, for all absorbing $i \neq s$, are evident from the definitions. To verify the remaining equations, we argue as follows. Let us consider a transient state $i$ and let $A$ be the event that state $s$ is eventually reached. We have

$$a_i = P(A \mid X_0 = i)$$

$$= \sum_{j=1}^{m} P(A \mid X_0 = i, X_1 = j)P(X_1 = j \mid X_0 = i) \quad \text{(total probability thm.)}$$

$$= \sum_{j=1}^{m} P(A \mid X_1 = j)p_{ij} \quad \text{(Markov property)}$$

$$= \sum_{j=1}^{m} a_jp_{ij}.$$

The uniqueness property of the solution to the absorption probability equations requires a separate argument, which is given in the end-of-chapter problems.

The next example illustrates how we can use the preceding method to calculate the probability of entering a given recurrent class (rather than a given absorbing state).
Example 7.10. Consider the Markov chain shown in Fig. 7.17(a). Note that there are two recurrent classes, namely \{1\} and \{4, 5\}. We would like to calculate the probability that the state eventually enters the recurrent class \{4, 5\} starting from one of the transient states. For the purposes of this problem, the possible transitions within the recurrent class \{4, 5\} are immaterial. We can therefore lump the states in this recurrent class and treat them as a single absorbing state (call it state 6), as in Fig. 7.17(b). It then suffices to compute the probability of eventually entering state 6 in this new chain.

![Transition probability graph](image)

Figure 7.17: (a) Transition probability graph in Example 7.10. (b) A new graph in which states 4 and 5 have been lumped into the absorbing state 6.

The probabilities of eventually reaching state 6, starting from the transient states 2 and 3, satisfy the following equations:

\[
a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6.
\]

\[
a_3 = 0.2a_2 + 0.8a_6.
\]

Using the facts \(a_1 = 0\) and \(a_6 = 1\), we obtain

\[
a_2 = 0.3a_2 + 0.4a_3 + 0.1.
\]

\[
a_3 = 0.2a_2 + 0.8.
\]

This is a system of two equations in the two unknowns \(a_2\) and \(a_3\), which can be readily solved to yield \(a_2 = 21/31\) and \(a_3 = 29/31\).

Example 7.11. Gambler's Ruin. A gambler wins $1 at each round with probability \(p\), and loses $1, with probability \(1 - p\). Different rounds are assumed
independent. The gambler plays continuously until he either accumulates a target amount of \$m$, or loses all his money. What is the probability of eventually accumulating the target amount (winning) or of losing his fortune?

We introduce the Markov chain shown in Fig. 7.18 whose state $i$ represents the gambler’s wealth at the beginning of a round. The states $i = 0$ and $i = m$ correspond to losing and winning, respectively.

All states are transient, except for the winning and losing states which are absorbing. Thus, the problem amounts to finding the probabilities of absorption at each one of these two absorbing states. Of course, these absorption probabilities depend on the initial state $i$.

![Transition probability graph for the gambler's ruin problem](image)

**Figure 7.18:** Transition probability graph for the gambler’s ruin problem (Example 7.11). Here $m = 4$.

Let us set $s = m$ in which case the absorption probability $a_i$ is the probability of winning, starting from state $i$. These probabilities satisfy

\[
\begin{align*}
  a_0 &= 0, \\
  a_i &= (1 - p)a_{i-1} + pa_{i+1}, & i &= 1, \ldots, m - 1, \\
  a_m &= 1.
\end{align*}
\]

These equations can be solved in a variety of ways. It turns out there is an elegant method that leads to a nice closed form solution.

Let us write the equations for the $a_i$ as

\[
(1 - p)(a_i - a_{i-1}) = p(a_{i+1} - a_i), \quad i = 1, \ldots, m - 1.
\]

Then, by denoting

\[
\delta_i = a_{i+1} - a_i, \quad i = 0, \ldots, m - 1,
\]

and

\[
\rho = \frac{1 - p}{p},
\]

the equations are written as

\[
\delta_i = \rho \delta_{i-1}, \quad i = 1, \ldots, m - 1.
\]

from which we obtain

\[
\delta_i = \rho^i \delta_0. \quad i = 1, \ldots, m - 1.
\]
This, together with the equation \( \delta_0 + \delta_1 + \cdots + \delta_{m-1} = a_m - a_0 = 1 \), implies that

\[
(1 + \rho + \cdots + \rho^{m-1})\delta_0 = 1,
\]

and

\[
\delta_0 = \frac{1}{1 + \rho + \cdots + \rho^{m-1}}.
\]

Since \( a_0 = 0 \) and \( a_{i+1} = a_i + \delta_i \), the probability \( a_i \) of winning starting from a fortune \( i \) is equal to

\[
a_i = \delta_0 + \delta_1 + \cdots + \delta_{i-1}
= (1 + \rho + \cdots + \rho^{i-1})\delta_0
= \frac{1 + \rho + \cdots + \rho^{i-1}}{1 + \rho + \cdots + \rho^{m-1}},
\]

which simplifies to

\[
a_i = \begin{cases} 
\frac{1 - \rho^i}{1 - \rho^m}, & \text{if } \rho \neq 1, \\
n/m, & \text{if } \rho = 1.
\end{cases}
\]

The solution reveals that if \( \rho > 1 \), which corresponds to \( p < 1/2 \) and unfavorable odds for the gambler, the probability of winning approaches 0 as \( m \to \infty \), for any fixed initial fortune. This suggests that if you aim for a large profit under unfavorable odds, financial ruin is almost certain.

**Expected Time to Absorption**

We now turn our attention to the expected number of steps until a recurrent state is entered (an event that we refer to as "absorption"), starting from a particular transient state. For any state \( i \), we denote

\[
\mu_i = \mathbb{E}[\text{number of transitions until absorption, starting from } i]
= \mathbb{E}[\min\{n \geq 0 \mid X_n \text{ is recurrent}\} \mid X_0 = i].
\]

Note that if \( i \) is recurrent, then \( \mu_i = 0 \) according to this definition.

We can derive equations for the \( \mu_i \) by using the total expectation theorem. We argue that the time to absorption starting from a transient state \( i \) is equal to 1 plus the expected time to absorption starting from the next state, which is \( j \) with probability \( p_{ij} \). We then obtain a system of linear equations, stated below, which is known to have a unique solution (see Problem 33 for the main idea).
Equations for the Expected Time to Absorption

The expected times to absorption, $\mu_1, \ldots, \mu_m$, are the unique solution to the equations

\[
\begin{align*}
\mu_i &= 0, & \text{for all recurrent states } i, \\
\mu_i &= 1 + \sum_{j=1}^{m} p_{ij} \mu_j, & \text{for all transient states } i.
\end{align*}
\]

Example 7.12. Spiders and Fly. Consider the spiders-and-fly model of Example 7.2. This corresponds to the Markov chain shown in Fig. 7.19. The states correspond to possible fly positions, and the absorbing states 1 and $m$ correspond to capture by a spider.

Let us calculate the expected number of steps until the fly is captured. We have

\[
\mu_1 = \mu_m = 0,
\]

and

\[
\mu_i = 1 + 0.3\mu_{i-1} + 0.4\mu_i + 0.3\mu_{i+1}. \quad \text{for } i = 2, \ldots, m - 1.
\]

We can solve these equations in a variety of ways, such as for example by successive substitution. As an illustration, let $m = 4$, in which case, the equations reduce to

\[
\begin{align*}
\mu_2 &= 1 + 0.4\mu_2 + 0.3\mu_3, \\
\mu_3 &= 1 + 0.3\mu_2 + 0.4\mu_3.
\end{align*}
\]

The first equation yields $\mu_2 = (1/0.6) + (1/2)\mu_3$, which we can substitute in the second equation and solve for $\mu_3$. We obtain $\mu_3 = 10/3$ and by substitution again. $\mu_2 = 10/3$.

![Transition probability graph in Example 7.12.](image)

Mean First Passage and Recurrence Times

The idea used to calculate the expected time to absorption can also be used to calculate the expected time to reach a particular recurrent state, starting
from any other state. For simplicity, we consider a Markov chain with a single recurrent class. We focus on a special recurrent state \( s \), and we denote by \( t_i \) the mean first passage time from state \( i \) to state \( s \), defined by

\[
t_i = \mathbb{E}[\text{number of transitions to reach } s \text{ for the first time, starting from } i]
= \mathbb{E}[\min\{n \geq 0 \mid X_n = s\} \mid X_0 = i].
\]

The transitions out of state \( s \) are irrelevant to the calculation of the mean first passage times. We may thus consider a new Markov chain which is identical to the original, except that the special state \( s \) is converted into an absorbing state (by setting \( p_{ss} = 1 \), and \( p_{sj} = 0 \) for all \( j \neq s \)). With this transformation, all states other than \( s \) become transient. We then compute \( t_i \) as the expected number of steps to absorption starting from \( i \), using the formulas given earlier in this section. We have

\[
t_i = 1 + \sum_{j=1}^{m} p_{ij}t_j, \quad \text{for all } i \neq s,
\]

\[
t_s = 0.
\]

This system of linear equations can be solved for the unknowns \( t_i \), and has a unique solution (see the end-of-chapter problems).

The above equations give the expected time to reach the special state \( s \) starting from any other state. We may also want to calculate the mean recurrence time of the special state \( s \), which is defined as

\[
t_s^* = \mathbb{E}[\text{number of transitions up to the first return to } s, \text{ starting from } s]
= \mathbb{E}[\min\{n \geq 1 \mid X_n = s\} \mid X_0 = s].
\]

We can obtain \( t_s^* \), once we have the first passage times \( t_i \), by using the equation

\[
t_s^* = 1 + \sum_{j=1}^{m} p_{sj}t_j.
\]

To justify this equation, we argue that the time to return to \( s \), starting from \( s \), is equal to 1 plus the expected time to reach \( s \) from the next state, which is \( j \) with probability \( p_{sj} \). We then apply the total expectation theorem.

**Example 7.13.** Consider the “up-to-date”–“behind” model of Example 7.1. States 1 and 2 correspond to being up-to-date and being behind, respectively, and the transition probabilities are

\[
p_{11} = 0.8, \quad p_{12} = 0.2,
\]

\[
p_{21} = 0.6, \quad p_{22} = 0.4.
\]
Let us focus on state $s = 1$ and calculate the mean first passage time to state 1. starting from state 2. We have $t_1 = 0$ and

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2.$$  

from which

$$t_2 = \frac{1}{0.6} = \frac{5}{3}.$$  

The mean recurrence time to state 1 is given by

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0.2 \cdot \frac{5}{3} = \frac{4}{3}.$$  

**Equations for Mean First Passage and Recurrence Times**

Consider a Markov chain with a single recurrent class, and let $s$ be a particular recurrent state.

- The mean first passage times $t_i$ to reach state $s$ starting from $i$, are the unique solution to the system of equations

  $$t_s = 0, \quad t_i = 1 + \sum_{j=1}^{m} p_{ij}t_j, \quad \text{for all } i \neq s.$$  

- The mean recurrence time $t_s^*$ of state $s$ is given by

  $$t_s^* = 1 + \sum_{j=1}^{m} p_{sj}t_j.$$  

7.5 CONTINUOUS-TIME MARKOV CHAINS

In the Markov chain models that we have considered so far, we have assumed that the transitions between states take unit time. In this section, we consider a related class of models that evolve in continuous time and can be used to study systems involving continuous-time arrival processes. Examples are distribution centers or nodes in communication networks where some events of interest (e.g., arrivals of orders or of new calls) are described in terms of Poisson processes.

As before, we will consider a process that involves transitions from one state to the next, according to given transition probabilities, but we will model the times spent between transitions as continuous random variables. We will still assume that the number of states is finite and, in the absence of a statement to the contrary, we will let the state space be the set $\mathcal{S} = \{1, \ldots, m\}$. 
To describe the process, we introduce certain random variables of interest:

- $X_n$: the state right after the $n$th transition;
- $Y_n$: the time of the $n$th transition;
- $T_n$: the time elapsed between the $(n - 1)$st and the $n$th transition.

For completeness, we denote by $X_0$ the initial state, and we let $Y_0 = 0$. We also introduce some assumptions.

**Continuous-Time Markov Chain Assumptions**

- If the current state is $i$, the time until the next transition is exponentially distributed with a given parameter $\nu_i$, independent of the past history of the process and of the next state.
- If the current state is $i$, the next state will be $j$ with a given probability $p_{ij}$, independent of the past history of the process and of the time until the next transition.

The above assumptions are a complete description of the process and provide an unambiguous method for simulating it: given that we just entered state $i$, we remain at state $i$ for a time that is exponentially distributed with parameter $\nu_i$, and then move to a next state $j$ according to the transition probabilities $p_{ij}$. As an immediate consequence, the sequence of states $X_n$ obtained after successive transitions is a discrete-time Markov chain, with transition probabilities $P_{ij}$, called the **embedded** Markov chain. In mathematical terms, our assumptions can be formulated as follows. Let

$$A = \{T_1 = t_1, \ldots, T_n = t_n, X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\}$$

be an event that captures the history of the process until the $n$th transition. We then have

$$P(X_{n+1} = j, T_{n+1} \geq t \mid A) = P(X_{n+1} = j, T_{n+1} \geq t \mid X_n = i)$$

$$= P(X_{n+1} = j \mid X_n = i)P(T_{n+1} \geq t \mid X_n = i)$$

$$= p_{ij}e^{-\nu_i t}, \quad \text{for all } t \geq 0.$$

The expected time to the next transition is

$$E[T_{n+1} \mid X_n = i] = \int_0^\infty \nu_i e^{-\nu_i \tau} \, d\tau = \frac{1}{\nu_i},$$

so we can interpret $\nu_i$ as the average number of transitions out of state $i$, per unit time spent at state $i$. Consequently, the parameter $\nu_i$ is called the transition
rate out of state \( i \). Since only a fraction \( p_{ij} \) of the transitions out of state \( i \) will lead to state \( j \), we may also view

\[
q_{ij} = \nu_i p_{ij}
\]

as the average number of transitions from \( i \) to \( j \), per unit time spent at \( i \). Accordingly, we call \( q_{ij} \) the transition rate from \( i \) to \( j \). Note that given the transition rates \( q_{ij} \), one can obtain the transition rates \( \nu_i \) using the formula

\[
\nu_i = \sum_{j=1}^{m} q_{ij},
\]

and the transition probabilities using the formula

\[
p_{ij} = \frac{q_{ij}}{\nu_i}.
\]

Note that the model allows for self-transitions, from a state back to itself, which can indeed happen if a self-transition probability \( p_{ii} \) is nonzero. However, such self-transitions have no observable effects: because of the memorylessness of the exponential distribution, the remaining time until the next transition is the same, irrespective of whether a self-transition just occurred or not. For this reason, we can ignore self-transitions and we will henceforth assume that

\[
p_{ii} = q_{ii} = 0, \quad \text{for all } i.
\]

**Example 7.14.** A machine, once in production mode, operates continuously until an alarm signal is generated. The time up to the alarm signal is an exponential random variable with parameter 1. Subsequent to the alarm signal, the machine is tested for an exponentially distributed amount of time with parameter 5. The test results are positive, with probability 1/2, in which case the machine returns to production mode, or negative, with probability 1/2, in which case the machine is taken for repair. The duration of the repair is exponentially distributed with parameter 3. We assume that the above mentioned random variables are all independent and also independent of the test results.

Let states 1, 2, and 3, correspond to production mode, testing, and repair, respectively. The transition rates are \( \nu_1 = 1, \nu_2 = 5 \), and \( \nu_3 = 3 \). The transition probabilities and the transition rates are given by the following two matrices:

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
1 & 0 & 0
\end{bmatrix}, \quad Q = \begin{bmatrix}
0 & 1 & 0 \\
5/2 & 0 & 5/2 \\
3 & 0 & 0
\end{bmatrix}.
\]

See Fig. 7.20 for an illustration.
Figure 7.20: Illustration of the Markov chain in Example 7.14. The quantities indicated next to each arc are the transition rates \( q_{ij} \).

We finally note that the continuous-time process we have described has a Markov property similar to its discrete-time counterpart: the future is independent of the past, given the present. To see this, denote by \( X(t) \) the state of a continuous-time Markov chain at time \( t \geq 0 \), and note that it stays constant between transitions.\(^\dagger\) Let us recall the memorylessness property of the exponential distribution, which in our context implies that for any time \( t \) between the \( n \)th and \( (n + 1) \)st transition times \( Y_n \) and \( Y_{n+1} \), the additional time \( Y_{n+1} - t \) until the next transition is independent of the time \( t - Y_n \) that the system has been in the current state. It follows that for any time \( t \), and given the present state \( X(t) \), the future of the process [the random variables \( X(\tau) \) for \( \tau > t \)], is independent of the past [the random variables \( X(\tau) \) for \( \tau < t \)].

**Approximation by a Discrete-Time Markov Chain**

We now elaborate on the relation between a continuous-time Markov chain and a corresponding discrete-time version. This relation will lead to an alternative description of a continuous-time Markov chain, and to a set of balance equations characterizing the steady-state behavior.

Let us fix a small positive number \( \delta \) and consider the discrete-time Markov chain \( Z_n \) that is obtained by observing \( X(t) \) every \( \delta \) time units:

\[
Z_n = X(n\delta), \quad n = 0, 1, \ldots
\]

The fact that \( Z_n \) is a Markov chain (the future is independent from the past, given the present) follows from the Markov property of \( X(t) \). We will use the notation \( \bar{p}_{ij} \) to describe the transition probabilities of \( Z_n \).

Given that \( Z_n = i \), there is a probability approximately equal to \( \nu_i \delta \) that there is a transition between times \( n\delta \) and \( (n + 1)\delta \), and in that case there is a

\(^\dagger\) If a transition takes place at time \( t \), the notation \( X(t) \) is ambiguous. A common convention is to let \( X(t) \) refer to the state right after the transition, so that \( X(Y_n) \) is the same as \( X_n \).
further probability \( p_{ij} \) that the next state is \( j \). Therefore,

\[
\bar{p}_{ij} = P(Z_{n+1} = j \mid Z_n = i) = \nu_i p_{ij} \delta + o(\delta) = q_{ij} \delta + o(\delta), \quad \text{if } j \neq i,
\]

where \( o(\delta) \) is a term that is negligible compared to \( \delta \), as \( \delta \) gets smaller. The probability of remaining at \( i \) [i.e., no transition occurs between times \( n\delta \) and \((n + 1)\delta \)] is

\[
\bar{p}_{ii} = P(Z_{n+1} = i \mid Z_n = i) = 1 - \sum_{j \neq i} \bar{p}_{ij}.
\]

This gives rise to the following alternative description.†

### Alternative Description of a Continuous-Time Markov Chain

Given the current state \( i \) of a continuous-time Markov chain, and for any \( j \neq i \), the state \( \delta \) time units later is equal to \( j \) with probability

\[
q_{ij} \delta + o(\delta),
\]

independent of the past history of the process.

---

**Example 7.14 (continued).** Neglecting \( o(\delta) \) terms, the transition probability matrix for the corresponding discrete-time Markov chain \( Z_n \) is

\[
\begin{bmatrix}
1 - \delta & \delta & 0 \\
5\delta/2 & 1 - 5\delta & 5\delta/2 \\
3\delta & 0 & 1 - 3\delta
\end{bmatrix}.
\]

**Example 7.15. Queueing.** Packets arrive at a node of a communication network according to a Poisson process with rate \( \lambda \). The packets are stored at a buffer with room for up to \( m \) packets, and are then transmitted one at a time. However, if a packet finds a full buffer upon arrival, it is discarded. The time required to transmit a packet is exponentially distributed with parameter \( \mu \). The transmission times of different packets are independent and are also independent from all the interarrival times.

We will model this system using a continuous-time Markov chain with state \( X(t) \) equal to the number of packets in the system at time \( t \) [if \( X(t) > 0 \), then

---

† Our argument so far shows that a continuous-time Markov chain satisfies this alternative description. Conversely, it can be shown that if we start with this alternative description, the time until a transition out of state \( i \) is an exponential random variable with parameter \( \nu_i = \sum_j q_{ij} \). Furthermore, given that such a transition has just occurred, the next state is \( j \) with probability \( q_{ij}/\nu_i = p_{ij} \). This establishes that the alternative description is equivalent to the original one.
$X(t)$ – 1 packets are waiting in the queue and one packet is under transmission]. The state increases by one when a new packet arrives and decreases by one when an existing packet departs. To show that $X(t)$ is indeed a Markov chain, we verify that we have the property specified in the above alternative description, and at the same time identify the transition rates $q_{ij}$.

Consider first the case where the system is empty, i.e., the state $X(t)$ is equal to 0. A transition out of state 0 can only occur if there is a new arrival, in which case the state becomes equal to 1. Since arrivals are Poisson, we have

$$P(X(t + \delta) = 1 \mid X(t) = 0) = \lambda \delta + o(\delta),$$

and

$$q_{0j} = \begin{cases} 
\lambda, & \text{if } j = 1, \\
0, & \text{otherwise.}
\end{cases}$$

Consider next the case where the system is full, i.e., the state $X(t)$ is equal to $m$. A transition out of state $m$ will occur upon the completion of the current packet transmission, at which point the state will become $m-1$. Since the duration of a transmission is exponential (and in particular, memoryless), we have

$$P(X(t + \delta) = m - 1 \mid X(t) = m) = \mu \delta + o(\delta),$$

and

$$q_{mj} = \begin{cases} 
\mu, & \text{if } j = m - 1, \\
0, & \text{otherwise.}
\end{cases}$$

Consider finally the case where $X(t)$ is equal to some intermediate state $i$, with $0 < i < m$. During the next $\delta$ time units, there is a probability $\lambda \delta + o(\delta)$ of a new packet arrival, which will bring the state to $i + 1$, and a probability $\mu \delta + o(\delta)$ that a packet transmission is completed, which will bring the state to $i - 1$. [The probability of both an arrival and a departure within an interval of length $\delta$ is of the order of $\delta^2$ and can be neglected, as is the case with other $o(\delta)$ terms.] Hence,

$$P(X(t + \delta) = i - 1 \mid X(t) = i) = \mu \delta + o(\delta),$$

$$P(X(t + \delta) = i + 1 \mid X(t) = i) = \lambda \delta + o(\delta),$$

and

$$q_{ij} = \begin{cases} 
\lambda, & \text{if } j = i + 1, \\
\mu, & \text{if } j = i - 1, \\
0, & \text{otherwise,}
\end{cases} \quad \text{for } i = 1, 2, \ldots, m - 1,$$

see Fig. 7.21.

![Figure 7.21: Transition graph in Example 7.15.](image-url)
Sec. 7.5 Continuous-Time Markov Chains

Steady-State Behavior

We now turn our attention to the long-term behavior of a continuous-time Markov chain and focus on the state occupancy probabilities $P(X(t) = i)$, in the limit as $t$ gets large. We approach this problem by studying the steady-state probabilities of the corresponding discrete-time chain $Z_n$.

Since $Z_n = X(n\delta)$, it is clear that the limit $\pi_j$ of $P(Z_n = j \mid Z_0 = i)$, if it exists, is the same as the limit of $P(X(t) = j \mid X(0) = i)$. It therefore suffices to consider the steady-state probabilities associated with $Z_n$. Reasoning as in the discrete-time case, we see that for the steady-state probabilities to be independent of the initial state, we need the chain $Z_n$ to have a single recurrent class, which we will henceforth assume. We also note that the Markov chain $Z_n$ is automatically aperiodic. This is because the self-transition probabilities are of the form

$$\bar{p}_{ii} = 1 - \delta \sum_{j \neq i} q_{ij} + o(\delta),$$

which is positive when $\delta$ is small, and because chains with nonzero self-transition probabilities are always aperiodic.

The balance equations for the chain $Z_n$ are of the form

$$\pi_j = \sum_{k=1}^{m} \pi_k \bar{p}_{kj} \quad \text{for all } j,$$

or

$$\pi_j = \pi_j \bar{p}_{jj} + \sum_{k \neq j} \pi_k \bar{p}_{kj}$$

$$= \pi_j \left( 1 - \delta \sum_{k \neq j} q_{jk} + o(\delta) \right) + \sum_{k \neq j} \pi_k (q_{kj} \delta + o(\delta)).$$

We cancel out the term $\pi_j$ that appears on both sides of the equation, divide by $\delta$, and take the limit as $\delta$ decreases to zero, to obtain the balance equations

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k q_{kj}.$$

We can now invoke the Steady-State Convergence Theorem for the chain $Z_n$ to obtain the following.

**Steady-State Convergence Theorem**

Consider a continuous-time Markov chain with a single recurrent class. Then, the states $j$ are associated with steady-state probabilities $\pi_j$ that have the following properties.
(a) For each $j$, we have
\[
\lim_{t \to \infty} P(X(t) = j \mid X(0) = i) = \pi_j, \quad \text{for all } i.
\]

(b) The $\pi_j$ are the unique solution to the system of equations below:
\[
\begin{align*}
\pi_j \sum_{k \neq j} q_{jk} &= \sum_{k \neq j} \pi_k q_{kj}, & j = 1, \ldots, m, \\
1 &= \sum_{k=1}^{m} \pi_k.
\end{align*}
\]

(c) We have
\[
\begin{align*}
\pi_j &= 0, \quad \text{for all transient states } j, \\
\pi_j &> 0, \quad \text{for all recurrent states } j.
\end{align*}
\]

To interpret the balance equations, we view $\pi_j$ as the expected long-term fraction of time the process spends in state $j$. It follows that $\pi_k q_{kj}$ can be viewed as the expected frequency of transitions from $k$ to $j$ (expected number of transitions from $k$ to $j$ per unit time). It is seen therefore that the balance equations express the intuitive fact that the frequency of transitions out of state $j$ (the left-hand side term $\pi_j \sum_{k \neq j} q_{jk}$) is equal to the frequency of transitions into state $j$ (the right-hand side term $\sum_{k \neq j} \pi_k q_{kj}$).

Example 7.14 (continued). The balance and normalization equations for this example are
\[
\begin{align*}
\pi_1 &= \frac{5}{2} \pi_2 + 3 \pi_3, & 5 \pi_2 = \pi_1, & 3 \pi_3 = \frac{5}{2} \pi_2, \\
1 &= \pi_1 + \pi_2 + \pi_3.
\end{align*}
\]
As in the discrete-time case, one of these equations is redundant, e.g., the third equation can be obtained from the first two. Still, there is a unique solution:
\[
\begin{align*}
\pi_1 &= \frac{30}{41}, & \pi_2 &= \frac{6}{41}, & \pi_3 &= \frac{5}{41}.
\end{align*}
\]
Thus, for example, if we let the process run for a long time, $X(t)$ will be at state 1 with probability 30/41, independent of the initial state.

The steady-state probabilities $\pi_j$ are to be distinguished from the steady-state probabilities $\overline{\pi}_j$ of the embedded Markov chain $X_n$. Indeed, the balance and normalization equations for the embedded Markov chain are
\[
\begin{align*}
\overline{\pi}_1 &= \frac{1}{2} \overline{\pi}_2 + \overline{\pi}_3, & \overline{\pi}_2 &= \overline{\pi}_1, & \overline{\pi}_3 &= \frac{1}{2} \overline{\pi}_2,
\end{align*}
\]
1 = \pi_1 + \pi_2 + \pi_3,

yielding the solution

\pi_1 = \frac{2}{5}, \quad \pi_2 = \frac{2}{5}, \quad \pi_3 = \frac{1}{5}.

To interpret the probabilities \pi_j, we can say, for example, that if we let the process run for a long time, the expected fraction of transitions that lead to state 1 is equal to 2/5.

Note that even though \pi_1 = \pi_2 (that is, there are about as many transitions into state 1 as there are transitions into state 2), we have \pi_1 > \pi_2. The reason is that the process tends to spend more time during a typical visit to state 1 than during a typical visit to state 2. Hence, at a given time t, the process \(X(t)\) is more likely to be found at state 1. This situation is typical, and the two sets of steady-state probabilities (\(\pi_j\) and \(\pi_j\)) are generically different. The main exception arises in the special case where the transition rates \(\nu_i\) are the same for all \(i\); see the end-of-chapter problems.

Birth-Death Processes

As in the discrete-time case, the states in a birth-death process are linearly arranged and transitions can only occur to a neighboring state, or else leave the state unchanged; formally, we have

\[q_{ij} = 0, \quad \text{for } |i - j| > 1.\]

In a birth-death process, the long-term expected frequencies of transitions from \(i\) to \(j\) and of transitions from \(j\) to \(i\) must be the same, leading to the local balance equations

\[\pi_j q_{ji} = \pi_i q_{ij}, \quad \text{for all } i, j.\]

The local balance equations have the same structure as in the discrete-time case, leading to closed-form formulas for the steady-state probabilities.

Example 7.15 (continued). The local balance equations take the form

\[\pi_i \lambda = \pi_{i+1} \mu, \quad i = 0, 1, \ldots, m - 1,\]

and we obtain \(\pi_{i+1} = \rho \pi_i\), where \(\rho = \lambda/\mu\). Thus, we have \(\pi_i = \rho^i \pi_0\) for all \(i\). The normalization equation \(1 = \sum_{i=0}^{m} \pi_i\) yields

\[1 = \pi_0 \sum_{i=0}^{m} \rho^i,\]

and the steady-state probabilities are

\[\pi_i = \frac{\rho^i}{1 + \rho + \cdots + \rho^n}. \quad i = 0, 1, \ldots, m.\]
7.6 SUMMARY AND DISCUSSION

In this chapter, we have introduced Markov chain models with a finite number of states. In a discrete-time Markov chain, transitions occur at integer times according to given transition probabilities $p_{ij}$. The crucial property that distinguishes Markov chains from general random processes is that the transition probabilities $p_{ij}$ apply each time that the state is equal to $i$, independent of the previous values of the state. Thus, given the present, the future of the process is independent of the past.

Coming up with a suitable Markov chain model of a given physical situation is to some extent an art. In general, we need to introduce a rich enough set of states so that the current state summarizes whatever information from the history of the process is relevant to its future evolution. Subject to this requirement, we usually aim at a model that does not involve more states than necessary.

Given a Markov chain model, there are several questions of interest.

(a) Questions referring to the statistics of the process over a finite time horizon. We have seen that we can calculate the probability that the process follows a particular path by multiplying the transition probabilities along the path. The probability of a more general event can be obtained by adding the probabilities of the various paths that lead to the occurrence of the event. In some cases, we can exploit the Markov property to avoid listing each and every path that corresponds to a particular event. A prominent example is the recursive calculation of the $n$-step transition probabilities, using the Chapman-Kolmogorov equations.

(b) Questions referring to the steady-state behavior of the Markov chain. To address such questions, we classified the states of a Markov chain as transient and recurrent. We discussed how the recurrent states can be divided into disjoint recurrent classes, so that each state in a recurrent class is accessible from every other state in the same class. We also distinguished between periodic and aperiodic recurrent classes. The central result of Markov chain theory is that if a chain consists of a single aperiodic recurrent class, plus possibly some transient states, the probability $r_{ij}(n)$ that the state is equal to some $j$ converges, as time goes to infinity, to a steady-state probability $\pi_j$, which does not depend on the initial state $i$. In other words, the identity of the initial state has no bearing on the statistics of $X_n$ when $n$ is very large. The steady-state probabilities can be found by solving a system of linear equations, consisting of the balance equations and the normalization equation $\sum_j \pi_j = 1$.

(c) Questions referring to the transient behavior of a Markov chain. We discussed the absorption probabilities (the probability that the state eventually enters a given recurrent class, given that it starts at a given transient state), and the mean first passage times (the expected time until a particular recurrent state is entered, assuming that the chain has a single recurrent
class). In both cases, we showed that the quantities of interest can be found by considering the unique solution to a system of linear equations.

We finally considered continuous-time Markov chains. In such models, given the current state, the next state is determined by the same mechanism as in discrete-time Markov chains. However, the time until the next transition is an exponentially distributed random variable, whose parameter depends only the current state. Continuous-time Markov chains are in many ways similar to their discrete-time counterparts. They have the same Markov property (the future is independent from the past, given the present). In fact, we can visualize a continuous-time Markov chain in terms of a related discrete-time Markov chain obtained by a fine discretization of the time axis. Because of this correspondence, the steady-state behaviors of continuous-time and discrete-time Markov chains are similar: assuming that there is a single recurrent class, the occupancy probability of any particular state converges to a steady-state probability that does not depend on the initial state. These steady-state probabilities can be found by solving a suitable set of balance and normalization equations.
SECTION 7.1. Discrete-Time Markov Chains

Problem 1. The times between successive customer arrivals at a facility are independent and identically distributed random variables with the following PMF:

\[ p(k) = \begin{cases} 
0.2, & k=1, \\
0.3, & k=3, \\
0.5, & k=4, \\
0, & \text{otherwise}.
\end{cases} \]

Construct a four-state Markov chain model that describes the arrival process. In this model, one of the states should correspond to the times when an arrival occurs.

Problem 2. A mouse moves along a tiled corridor with \(2m\) tiles, where \(m > 1\). From each tile \(i \neq 1, 2m\), it moves to either tile \(i - 1\) or \(i + 1\) with equal probability. From tile 1 or tile \(2m\), it moves to tile 2 or \(2m - 1\), respectively, with probability 1. Each time the mouse moves to a tile \(i \leq m\) or \(i > m\), an electronic device outputs a signal \(L\) or \(R\), respectively. Can the generated sequence of signals \(L\) and \(R\) be described as a Markov chain with states \(L\) and \(R\)?

Problem 3. Consider the Markov chain in Example 7.2, for the case where \(m = 4\), as in Fig. 7.2, and assume that the process starts at any of the four states, with equal probability. Let \(Y_n = 1\) whenever the Markov chain is at state 1 or 2, and \(Y_n = 2\) whenever the Markov chain is at state 3 or 4. Is the process \(Y_n\) a Markov chain?

SECTION 7.2. Classification of States

Problem 4. A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is integer. When the spider and the fly land in the same position, the spider captures the fly.

(a) Construct a Markov chain that describes the relative location of the spider and fly.

(b) Identify the transient and recurrent states.

Problem 5. Consider a Markov chain with states 1, 2, \ldots, 9, and the following transition probabilities: \(p_{12} = p_{17} = 1/2\), \(p_{i(i+1)} = 1\) for \(i \neq 1, 6, 9\), and \(p_{61} = p_{91} = 1\). Is the recurrent class of the chain periodic or not?
Problem 6.* Existence of a recurrent state. Show that in a Markov chain at least one recurrent state must be accessible from any given state, i.e., for any \( i \), there is at least one recurrent \( j \) in the set \( A(i) \) of accessible states from \( i \).

Solution. Fix a state \( i \). If \( i \) is recurrent, then every \( j \in A(i) \) is also recurrent so we are done. Assume that \( i \) is transient. Then, there exists a state \( i_1 \in A(i) \) such that \( i \notin A(i_1) \). If \( i_1 \) is recurrent, then we have found a recurrent state that is accessible from \( i \), and we are done. Suppose now that \( i_1 \) is transient. Then, \( i_1 \notin A(i) \) because otherwise the assumptions \( i_1 \in A(i) \) and \( i \notin A(i_1) \) would yield \( i \notin A(i) \) and \( i \notin A(i) \), which is a contradiction. Since \( i_1 \) is transient, there exists some \( i_2 \) such that \( i_2 \in A(i_1) \) and \( i_1 \notin A(i_2) \). In particular, \( i_2 \notin A(i) \). If \( i_2 \) is recurrent, we are done. So, suppose that \( i_2 \) is transient. The same argument as before shows that \( i_2 \neq i_1 \). Furthermore, we must also have \( i_2 \neq i \). This is because if we had \( i_2 = i \), we would have \( i_1 \in A(i) = A(i_2) \), contradicting the assumption \( i_1 \notin A(i_2) \). Continuing this process, at the \( k \)th step, we will either obtain a recurrent state \( i_k \) which is accessible from \( i \), or else we will obtain a transient state \( i_k \) which is different than all the preceding states \( i, i_1, \ldots, i_{k-1} \). Since there is only a finite number of states, a recurrent state must ultimately be obtained.

Problem 7.* Consider a Markov chain with some transient and some recurrent states.

(a) Show that for some numbers \( c \) and \( \gamma \), with \( c > 0 \) and \( 0 < \gamma < 1 \), we have

\[
P(X_n \text{ transient} \mid X_0 = i) \leq c\gamma^n, \quad \text{for all } i \text{ and } n \geq 1.
\]

(b) Let \( T \) be the first time \( n \) at which \( X_n \) is recurrent. Show that such a time is certain to exist (i.e., the probability of the event that there exists a time \( n \) at which \( X_n \) is recurrent is equal to 1) and that \( E[T] < \infty \).

Solution. (a) For notational convenience, let

\[
q_i(n) = P(X_n \text{ transient} \mid X_0 = i).
\]

A recurrent state that is accessible from state \( i \) can be reached in at most \( m \) steps, where \( m \) is the number of states. Therefore, \( q_i(m) < 1 \). Let

\[
\beta = \max_{i=1, \ldots, m} q_i(m)
\]

and note that for all \( i \), we have \( q_i(m) \leq \beta < 1 \). If a recurrent state has not been reached by time \( m \), which happens with probability at most \( \beta \), the conditional probability that a recurrent state is not reached in the next \( m \) steps is at most \( \beta \) as well, which suggests that \( q_i(2m) \leq \beta^2 \). Indeed, conditioning on the possible values of \( X_m \), we obtain

\[
q_i(2m) = P(X_{2m} \text{ transient} \mid X_0 = i)
= \sum_{j \text{ transient}} P(X_{2m} \text{ transient} \mid X_m = j, X_0 = i) P(X_m = j \mid X_0 = i)
= \sum_{j \text{ transient}} P(X_{2m} \text{ transient} \mid X_m = j) P(X_m = j \mid X_0 = i)
= \sum_{j \text{ transient}} P(X_m \text{ transient} \mid X_0 = j) P(X_m = j \mid X_0 = i)
\]
\[ \leq \beta \sum_{j \text{ transient}} P(X_m = j \mid X_0 = i) = \beta P(X_m \text{ transient} \mid X_0 = i) \leq \beta^2. \]

Continuing similarly, we obtain

\[ q_i(km) \leq \beta^k, \quad \text{for all } i \text{ and } k \geq 1. \]

Let \( n \) be any positive integer, and let \( k \) be the integer such that \( km \leq n < (k + 1)m \). Then, we have

\[ q_i(n) \leq q_i(km) \leq \beta^k = \beta^{-1} (\beta^{1/m})^{(k+1)m} \leq \beta^{-1} (\beta^{1/m})^n. \]

Thus, the desired relation holds with \( c = \beta^{-1} \) and \( \gamma = \beta^{1/m} \).

(b) Let \( A \) be the event that the state never enters the set of recurrent states. Using the result from part (a), we have

\[ P(A) \leq P(X_n \text{ transient}) \leq c\gamma^n. \]

Since this is true for every \( n \) and since \( \gamma < 1 \), we must have \( P(A) = 0 \). This establishes that there is certainty (probability equal to 1) that there is a finite time \( T \) that a recurrent state is first entered. We then have

\[ E[T] = \sum_{n=1}^{\infty} nP(X_{n-1} \text{ transient}, X_n \text{ recurrent}) \leq \sum_{n=1}^{\infty} nP(X_{n-1} \text{ transient}) \leq \sum_{n=1}^{\infty} nc\gamma^{n-1} = \frac{c}{1 - \gamma} \sum_{n=1}^{\infty} n(1 - \gamma)\gamma^{n-1} = \frac{c}{(1 - \gamma)^2}, \]

where the last equality is obtained using the expression for the mean of the geometric distribution.

**Problem 8.** Recurrent states. Show that if a recurrent state is visited once, the probability that it will be visited again in the future is equal to 1 (and, therefore, the probability that it will be visited an infinite number of times is equal to 1). *Hint:* Modify the chain to make the recurrent state of interest the only recurrent state, and use the result of Problem 7(b).

**Solution.** Let \( s \) be a recurrent state, and suppose that \( s \) has been visited once. From then on, the only possible states are those in the same recurrence class as \( s \). Therefore,
without loss of generality, we can assume that there is a single recurrent class. Suppose that the current state is some \( i \neq s \). We want to show that \( s \) is guaranteed to be visited some time in the future.

Consider a new Markov chain in which the transitions out of state \( s \) are disabled, so that \( p_{ss} = 1 \). The transitions out of states \( i, \) for \( i \neq s \) are unaffected. Clearly, \( s \) is recurrent in the new chain. Furthermore, for any state \( i \neq s \), there is a positive probability path from \( i \) to \( s \) in the original chain (since \( s \) is recurrent in the original chain), and the same holds true in the new chain. Since \( i \) is not accessible from \( s \) in the new chain, it follows that every \( i \neq s \) in the new chain is transient. By the result of Problem 7(b), state \( s \) will be eventually visited by the new chain (with probability 1). But the original chain is identical to the new one until the time that \( s \) is first visited. Hence, state \( s \) is guaranteed to be eventually visited by the original chain \( s \). By repeating this argument, we see that \( s \) is guaranteed to be visited an infinite number of times (with probability 1).

**Problem 9.** Periodic classes. Consider a recurrent class \( R \). Show that exactly one of the following two alternatives must hold:

(i) The states in \( R \) can be grouped in \( d > 1 \) disjoint subsets \( S_1, \ldots, S_d \), so that all transitions from \( S_k \) lead to \( S_{k+1} \), or to \( S_1 \) if \( k = d \). (In this case, \( R \) is periodic.)

(ii) There exists a time \( n \) such that \( r_{ij}(n) > 0 \) for all \( i, j \in R \). (In this case \( R \) is aperiodic.)

**Hint:** Fix a state \( i \) and let \( d \) be the greatest common divisor of the elements of the set \( Q = \{ n : r_{ii}(n) > 0 \} \). If \( d = 1 \), use the following fact from elementary number theory: if the positive integers \( \alpha_1, \alpha_2, \ldots \) have no common divisor other than 1, then every positive integer \( n \) outside a finite set can be expressed in the form \( n = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_t\alpha_t \) for some nonnegative integers \( k_1, \ldots, k_t \), and some \( t \geq 1 \).

**Solution.** Fix a state \( i \) and consider the set \( Q = \{ n : r_{ii}(n) > 0 \} \). Let \( d \) be the greatest common divisor of the elements of \( Q \). Consider first the case where \( d \neq 1 \). For \( k = 1, \ldots, d \), let \( S_k \) be the set of all states that are reachable from \( i \) in \( ld + k \) steps for some nonnegative integer \( l \). Suppose that \( s \in S_k \) and \( p_{ss'} > 0 \). Since \( s \in S_k \), we can reach \( s \) from \( i \) in \( ld + k \) steps for some \( l \), which implies that we can reach \( s' \) from \( i \) in \( ld + k + 1 \) steps. This shows that \( s' \in S_{k+1} \) if \( k < d \), and that \( s' \in S_1 \) if \( k = d \). It only remains to show that the sets \( S_1, \ldots, S_d \) are disjoint. Suppose, to derive a contradiction, that \( s \in S_k \) and \( s \in S_{k'} \) for some \( k \neq k' \). Let \( q \) be the length of some positive probability path from \( s \) to \( i \). Starting from \( i \), we can get to \( s \) in \( ld + k \) steps, and then return to \( i \) in \( q \) steps. Hence \( ld + k + q \) belongs to \( Q \), which implies that \( d \) divides \( k + q \). By the same argument, \( d \) must also divide \( k' + q \). Hence \( d \) divides \( k - k' \), which is a contradiction because \( 1 \leq |k - k'| \leq d - 1 \).

Consider next the case where \( d = 1 \). Let \( Q = \{ \alpha_1, \alpha_2, \ldots \} \). Since these are the possible lengths of positive probability paths that start and end at \( i \), it follows that any integer \( n \) of the form \( n = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_t\alpha_t \) is also in \( Q \). (To see this, use \( k_1 \) times a path of length \( \alpha_1 \), followed by using \( k_2 \) times a path of length \( \alpha_2 \), etc.) By the number-theoretic fact given in the hint, the set \( Q \) contains all but finitely many positive integers. Let \( n_i \) be such that

\[ r_{ii}(n) > 0, \quad \text{for all } n > n_i. \]

Fix some \( j \neq i \) and let \( q \) be the length of a shortest positive probability path from \( i \) to \( j \), so that \( q < m \), where \( m \) is the number of states. Consider some \( n \) that satisfies
We have so far established that at least one of the alternatives given in the problem statement must hold. To establish that they cannot hold simultaneously, note that the first alternative implies that \( r_{ii}(n) = 0 \) whenever \( n \) is not an integer multiple of \( d \), which is incompatible with the second alternative.

For completeness, we now provide a proof of the number-theoretic fact that was used in this problem. We start with the set of positive integers \( \alpha_1, \alpha_2, \ldots \), and assume that they have no common divisor other than 1. We define \( M \) as the set of all positive integers the form \( \sum_{i=1}^{t} k_i \alpha_i \), where the \( k_i \) are nonnegative integers. Note that this set is closed under addition (the sum of two elements of \( M \) is of the same form and must also belong to \( M \)). Let \( g \) be the smallest difference between two distinct elements of \( M \). Then, \( g \geq 1 \) and \( g \leq \alpha_i \) for all \( i \), since \( \alpha_i \) and \( 2\alpha_i \) both belong to \( M \).

Suppose that \( g > 1 \). Since the greatest common divisor of the \( \alpha_i \) is 1, there exists some \( \alpha_i^* \) which is not divisible by \( g \). We then have

\[ \alpha_i^* = \ell g + r, \]

for some positive integer \( \ell \), where the remainder \( r \) satisfies \( 0 < r < g \). Furthermore, in view of the definition of \( g \), there exist nonnegative integers \( k_1, k_1', \ldots, k_t, k_t' \) such that

\[ \sum_{i=1}^{t} k_i \alpha_i = \sum_{i=1}^{t} k_i' \alpha_i + g. \]

Multiplying this equation by \( \ell \) and using the equation \( \alpha_i^* = \ell g + r \), we obtain

\[ \sum_{i=1}^{t} (\ell k_i) \alpha_i = \sum_{i=1}^{t} (\ell k_i') \alpha_i + \ell g = \sum_{i=1}^{t} (\ell k_i') \alpha_i + \alpha_i^* - r. \]

This shows that there exist two numbers in the set \( M \), whose difference is equal to \( r \). Since \( 0 < r < g \), this contradicts our definition of \( g \) as the smallest possible difference. This contradiction establishes that \( g \) must be equal to 1.

Since \( g = 1 \), there exists some positive integer \( x \) such that \( x \in M \) and \( x + 1 \in M \). We will now show that every integer \( n \) larger than \( \alpha_1 x \) belongs to \( M \). Indeed, by dividing \( n \) by \( \alpha_1 \), we obtain \( n = k \alpha_1 + r \), where \( k \geq x \) and where the remainder \( r \) satisfies \( 0 \leq r < \alpha_1 \). We rewrite \( n \) in the form

\[ n = x(\alpha_1 - r) + (x + 1)r + (k - x)\alpha_1. \]

Since \( x, x + 1, \) and \( \alpha_1 \) all belong to \( M \), this shows that \( n \) is the sum of elements of \( M \) and must also belong to \( M \), as desired.

**SECTION 7.3. Steady-State Behavior**

**Problem 10.** Consider the two models of machine failure and repair in Example 7.3. Find conditions on \( b \) and \( r \) for the chain to have a single recurrent class which is
aperiodic and, under those conditions, find closed form expressions for the steady-state probabilities.

Problem 11. A professor gives tests that are hard, medium, or easy. If she gives a hard test, her next test will be either medium or easy, with equal probability. However, if she gives a medium or easy test, there is a 0.5 probability that her next test will be of the same difficulty, and a 0.25 probability for each of the other two levels of difficulty. Construct an appropriate Markov chain and find the steady-state probabilities.

Problem 12. Alvin likes to sail each Saturday to his cottage on a nearby island off the coast. Alvin is an avid fisherman, and enjoys fishing off his boat on the way to and from the island, as long as the weather is good. Unfortunately, the weather is good on the way to or from the island with probability \( p \), independent of what the weather was on any past trip (so the weather could be nice on the way to the island, but poor on the way back). Now, if the weather is nice, Alvin will take one of his \( n \) fishing rods for the trip, but if the weather is bad, he will not bring a fishing rod with him. We want to find the probability that on a given leg of the trip to or from the island the weather will be nice, but Alvin will not fish because all his fishing rods are at his other home.

(a) Formulate an appropriate Markov chain model with \( n + 1 \) states and find the steady-state probabilities.

(b) What is the steady-state probability that on a given trip, Alvin sails with nice weather but without a fishing rod?

Problem 13. Consider the Markov chain in Fig. 7.22. Let us refer to a transition that results in a state with a higher (respectively, lower) index as a birth (respectively, death). Calculate the following quantities, assuming that when we start observing the chain, it is already in steady-state.

![Figure 7.22: Transition probability graph for Problem 11.](image)

(a) For each state \( i \), the probability that the current state is \( i \).

(b) The probability that the first transition we observe is a birth.

(c) The probability that the first change of state we observe is a birth.

(d) The conditional probability that the process was in state 2 before the first transition that we observe, given that this transition was a birth.

(e) The conditional probability that the process was in state 2 before the first change of state that we observe, given that this change of state was a birth.
(f) The conditional probability that the first observed transition is a birth given that
it resulted in a change of state.

(g) The conditional probability that the first observed transition leads to state 2,
given that it resulted in a change of state.

**Problem 14.** Consider a Markov chain with given transition probabilities and with a
single recurrent class that is aperiodic. Assume that for \( n \geq 500 \), the \( n \)-step transition
probabilities are very close to the steady-state probabilities.

(a) Find an approximate formula for \( P(X_{1000} = j, X_{1001} = k, X_{2000} = l | X_0 = i) \).

(b) Find an approximate formula for \( P(X_{1000} = i | X_{1001} = j) \).

**Problem 15.** **Ehrenfest model of diffusion.** We have a total of \( n \) balls, some of
them black, some white. At each time step, we either do nothing, which happens with
probability \( \epsilon \), where \( 0 < \epsilon < 1 \), or we select a ball at random, so that each ball has
probability \((1 - \epsilon)/n > 0\) of being selected. In the latter case, we change the color of
the selected ball (if white it becomes black, and vice versa), and the process is repeated
indefinitely. What is the steady-state distribution of the number of white balls?

**Problem 16.** **Bernoulli-Laplace model of diffusion.** Each of two urns contains
\( m \) balls. Out of the total of the \( 2m \) balls, \( m \) are white and \( m \) are black. A ball is
simultaneously selected from each urn and moved to the other urn, and the process
is indefinitely repeated. What is the steady-state distribution of the number of white
balls in each urn?

**Problem 17.** Consider a Markov chain with two states denoted 1 and 2, and transition
probabilities

\[
\begin{align*}
p_{11} &= 1 - \alpha, & p_{12} &= \alpha, \\
p_{21} &= \beta, & p_{22} &= 1 - \beta,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are such that \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \).

(a) Show that the two states of the chain form a recurrent and aperiodic class.

(b) Use induction to show that for all \( n \), we have

\[
\begin{align*}
r_{11}(n) &= \frac{\beta}{\alpha + \beta} + \frac{\alpha(1 - \alpha - \beta)^n}{\alpha + \beta}, \\
r_{12}(n) &= \frac{\alpha}{\alpha + \beta} - \frac{\alpha(1 - \alpha - \beta)^n}{\alpha + \beta}, \\
r_{21}(n) &= \frac{\beta}{\alpha + \beta} - \frac{\beta(1 - \alpha - \beta)^n}{\alpha + \beta}, \\
r_{22}(n) &= \frac{\alpha}{\alpha + \beta} + \frac{\beta(1 - \alpha - \beta)^n}{\alpha + \beta}.
\end{align*}
\]

(c) What are the steady-state probabilities \( \pi_1 \) and \( \pi_2 \)?

**Problem 18.** The parking garage at MIT has installed a card-operated gate, which,
unfortunately, is vulnerable to absent-minded faculty and staff. In particular, in each
day, a car crashes the gate with probability \( p \), in which case a new gate must be
installed. Also a gate that has survived for \( m \) days must be replaced as a matter of
periodic maintenance. What is the long-term expected frequency of gate replacements?
Problem 19.* Steady-state convergence. Consider a Markov chain with a single recurrent class, and assume that there exists a time \( \bar{n} \) such that

\[ \tau_{ij}(\bar{n}) > 0, \]

for all \( i \) and all recurrent \( j \). (This is equivalent to assuming that the class is aperiodic.) We wish to show that for any \( i \) and \( j \), the limit

\[ \lim_{n \to \infty} \tau_{ij}(n) \]

exists and does not depend on \( i \). To derive this result, we need to show that the choice of the initial state has no long-term effect. To quantify this effect, we consider two different initial states \( i \) and \( k \), and consider two independent Markov chains, \( X_n \) and \( Y_n \), with the same transition probabilities and with \( X_0 = i \), \( Y_0 = k \). Let \( T = \min\{n \mid X_n = Y_n \} \) be the first time that the two chains enter the same state.

(a) Show that there exist positive constants \( c \) and \( \gamma < 1 \) such that

\[ P(T \geq n) \leq c\gamma^n. \]

(b) Show that if the states of the two chains became equal by time \( n \), their occupancy probabilities at time \( n \) are the same, that is,

\[ P(X_n = j \mid T \leq n) = P(Y_n = j \mid T \leq n). \]

(c) Show that \( |\tau_{ij}(n) - \tau_{kj}(n)| \leq c\gamma^n \), for all \( i, j, k \), and \( n \). Hint: Condition on the two events \( \{T > n\} \) and \( \{T \leq n\} \).

(d) Let \( q_j^+(n) = \max_i \tau_{ij}(n) \) and \( q_j^-(n) = \min_i \tau_{ij}(n) \). Show that

\[ q_j^-(n) \leq q_j^-(n + 1) \leq q_j^+(n + 1) \leq q_j^+(n), \quad \text{for all } n. \]

(e) Show that the sequence \( \tau_{ij}(n) \) converges to a limit that does not depend on \( i \). Hint: Combine the results of parts (c) and (d) to show that the two sequences \( q_j^-(n) \) and \( q_j^+(n) \) converge and have the same limit.

Solution. (a) The argument is similar to the one used to bound the PMF of the time until a recurrent state is entered (Problem 7). Let \( l \) be some recurrent state and let \( \beta = \min_i \tau_{il}(\bar{n}) > 0 \). No matter what is the current state of \( X_n \) and \( Y_n \), there is probability of at least \( \beta^2 \) that both chains are at state \( l \) after \( \bar{n} \) time steps. Thus,

\[ P(T > \bar{n}) \leq 1 - \beta^2. \]

Similarly,

\[ P(T > 2\bar{n}) = P(T > \bar{n}) P(T > 2\bar{n} \mid T > \bar{n}) \leq (1 - \beta^2)^2. \]

and

\[ P(T > k\bar{n}) \leq (1 - \beta^2)^k. \]
This implies that
\[ P(T \geq n) \leq c \gamma^n, \]
where \( \gamma = (1 - \beta^2)^{1/n}, \) and \( c = 1/(1 - \beta^2)^n. \)

(b) We condition on the possible values of \( T \) and on the common state \( l \) of the two chains at time \( T, \) and use the total probability theorem. We have

\[
P(X_n = j \mid T \leq n) = \sum_{t=0}^{n} \sum_{l=1}^{m} P(X_n = j \mid T = t, X_t = l) P(T = t, X_t = l \mid T \leq n)
= \sum_{t=0}^{n} \sum_{l=1}^{m} P(X_n = j \mid X_t = l) P(T = t, X_t = l \mid T \leq n)
= \sum_{t=0}^{n} \sum_{l=1}^{m} r_{lj}(n-t) P(T = t, X_t = l \mid T \leq n).
\]

Similarly,

\[
P(Y_n = j \mid T \leq n) = \sum_{t=0}^{n} \sum_{l=1}^{m} r_{lj}(n-t) P(T = t, Y_t = l \mid T \leq n).
\]

But the events \( \{T = t, X_t = l\} \) and \( \{T = t, Y_t = l\} \) are identical, and therefore have the same probability, which implies that \( P(X_n = j \mid T \leq n) = P(Y_n = j \mid T \leq n). \)

(c) We have

\[ r_{ij}(n) = P(X_n = j) = P(X_n = j \mid T \leq n) P(T \leq n) + P(X_n = j \mid T > n) P(T > n) \]
and

\[ r_{kj}(n) = P(Y_n = j) = P(Y_n = j \mid T \leq n) P(T \leq n) + P(Y_n = j \mid T > n) P(T > n). \]

By subtracting these two equations, using the result of part (b) to eliminate the first terms in their right-hand sides, and by taking the absolute value of both sides, we obtain

\[
|r_{ij}(n) - r_{kj}(n)| \leq |P(X_n = j \mid T > n) P(T > n) - P(Y_n = j \mid T > n) P(T > n)|
\leq P(T > n)
\leq c \gamma^n.
\]

(d) By conditioning on the state after the first transition, and using the total probability theorem, we have the following variant of the Chapman-Kolmogorov equation:

\[ r_{ij}(n + 1) = \sum_{k=1}^{m} p_{ik} r_{kj}(n). \]
Using this equation, we obtain

\[ q_j^+(n+1) = \max_i r_{ij}(n+1) = \max_i \sum_{k=1}^m p_{ik} r_{kj}(n) \leq \max_i \sum_{k=1}^m p_{ik} q_j^+(n) = q_j^+(n). \]

The inequality \( q_j^-(n) \leq q_j^-(n+1) \) is established by a symmetrical argument. The inequality \( q_j^-(n+1) \leq q_j^+(n+1) \) is a consequence of the definitions.

(e) The sequences \( q_j^-(n) \) and \( q_j^+(n) \) converge because they are monotonic. The inequality \( |r_{ij}(n) - r_{kj}(n)| \leq c\gamma^n \), for all \( i \) and \( k \), implies that \( q_j^+(n) - q_j^-(n) \leq c\gamma^n \). Taking the limit as \( n \to \infty \), we obtain that the limits of \( q_j^+(n) \) and \( q_j^-(n) \) are the same. Let \( \pi_j \) denote this common limit. Since \( q_j^-(n) \leq r_{ij}(n) \leq q_j^+(n) \), it follows that \( r_{ij}(n) \) also converges to \( \pi_j \), and the limit is independent of \( i \).

Problem 20.* Uniqueness of solutions to the balance equations. Consider a Markov chain with a single recurrent class, plus possibly some transient states.

(a) Assuming that the recurrent class is aperiodic, show that the balance equations together with the normalization equation have a unique nonnegative solution. Hint: Given a solution different from the steady-state probabilities, let it be the PMF of \( X_0 \) and consider what happens as time goes to infinity.

(b) Show that the uniqueness result of part (a) is also true when the recurrent class is periodic. Hint: Introduce self-transitions in the Markov chain, in a manner that results in an equivalent set of balance equations, and use the result of part (a).

Solution. (a) Let \( \pi_1, \ldots, \pi_m \) be the steady-state probabilities, that is, the limits of the \( r_{ij}(n) \). These satisfy the balance and normalization equations. Suppose that there is another nonnegative solution \( \overline{\pi}_1, \ldots, \overline{\pi}_m \). Let us initialize the Markov chain according to these probabilities, so that \( P(X_0 = j) = \overline{\pi}_j \) for all \( j \). Using the argument given in the text, we obtain \( P(X_n = j) = \overline{\pi}_j \) for all times. Thus,

\[
\overline{\pi}_j = \lim_{n \to \infty} P(X_n = j) = \lim_{n \to \infty} \sum_{k=1}^m \overline{\pi}_k r_{kj}(n) = \sum_{k=1}^m \overline{\pi}_k \pi_j = \pi_j.
\]

(b) Consider a new Markov chain, whose transition probabilities \( \overline{p}_{ij} \) are given by

\[
\overline{p}_{ii} = (1 - \alpha)p_{ii} + \alpha, \quad \overline{p}_{ij} = (1 - \alpha)p_{ij}, \quad j \neq i.
\]

Here, \( \alpha \) is a number satisfying \( 0 < \alpha < 1 \). The balance equations for the new Markov chain take the form

\[
\pi_j = \pi_j ((1 - \alpha)p_{jj} + \alpha) + \sum_{i \neq j} \pi_i (1 - \alpha)p_{ij},
\]
or

\[(1 - \alpha)\pi_j = (1 - \alpha) \sum_{i=1}^{m} \pi_i p_{ij}.
\]

These equations are equivalent to the balance equations for the original chain. Notice that the new chain is aperiodic, because self-transitions have positive probability. This establishes uniqueness of solutions for the new chain, and implies the same for the original chain.

**Problem 21.** *Expected long-term frequency interpretation.* Consider a Markov chain with a single recurrent class which is aperiodic. Show that

\[
\pi_j = \lim_{n \to \infty} \frac{v_{ij}(n)}{n}, \quad \text{for all } i, j = 1, \ldots, m,
\]

where the \(\pi_j\) are the steady-state probabilities, and \(v_{ij}(n)\) is the expected value of the number of visits to state \(j\) within the first \(n\) transitions, starting from state \(i\). *Hint:* Use the following fact from analysis. If a sequence \(a_n\) converges to a number \(a\), the sequence \(b_n = (1/n) \sum_{k=1}^{n} a_k\) also converges to \(a\).

**Solution.** We first assert that for all \(n, i, j\), we have

\[v_{ij}(n) = \sum_{k=1}^{n} r_{ij}(k).
\]

To see this, note that

\[v_{ij}(n) = \mathbb{E}\left[\sum_{k=1}^{n} I_k \mid X_0 = i\right],
\]

where \(I_k\) is the random variable that takes the value 1 if \(X_k = j\), and the value 0 otherwise, so that

\[\mathbb{E}[I_k \mid X_0 = i] = r_{ij}(k).
\]

Since

\[\frac{v_{ij}(n)}{n} = \frac{1}{n} \sum_{k=1}^{n} r_{ij}(k),
\]

and \(r_{ij}(k)\) converges to \(\pi_j\), it follows that \(v_{ij}(n)/n\) also converges to \(\pi_j\), which is the desired result.

For completeness, we also provide the proof of the fact given in the hint, and which was used in the last step of the above argument. Consider a sequence \(a_n\) that converges to some \(a\), and let \(b_n = (1/n) \sum_{k=1}^{n} a_k\). Fix some \(\epsilon > 0\). Since \(a_n\) converges to \(a\), there exists some \(n_0\) such that \(a_k \leq a + (\epsilon/2)\), for all \(k > n_0\). Let also \(c = \max_k a_k\). We then have

\[b_n = \frac{1}{n} \sum_{k=1}^{n_0} a_k + \frac{1}{n} \sum_{k=n_0+1}^{n} a_k \leq \frac{n_0}{n} c + \frac{n - n_0}{n} \left( a + \frac{\epsilon}{2} \right).
\]

The limit of the right-hand side, as \(n\) tends to infinity, is \(a + (\epsilon/2)\). Therefore, there exists some \(n_1\) such that \(b_n \leq a + \epsilon\), for every \(n \geq n_1\). By a symmetrical argument,
there exists some $n_2$ such that $b_n \geq a - \epsilon$, for every $n \geq n_2$. We have shown that for every $\epsilon > 0$, there exists some $n_3$ (namely, $n_3 = \max\{n_1, n_2\}$) such that $|b_n - a| \leq \epsilon$, for all $n \geq n_3$. This means that $b_n$ converges to $a$.

**Problem 22.** Doubly stochastic matrices. Consider a Markov chain with a single recurrent class which is aperiodic, and whose transition probability matrix is *doubly stochastic*, i.e., it has the property that the entries in any column (as well as in any row) add to unity, so that

$$\sum_{i=1}^{m} p_{ij} = 1, \quad j = 1, \ldots, m.$$  

(a) Show that the transition probability matrix of the chain in Example 7.7 is doubly stochastic.

(b) Show that the steady-state probabilities are

$$\pi_j = \frac{1}{m}, \quad j = 1, \ldots, m.$$  

(c) Suppose that the recurrent class of the chain is instead periodic. Show that $\pi_1 = \cdots = \pi_m = 1/m$ is the unique solution to the balance and normalization equations. Discuss your answer in the context of Example 7.7 for the case where $m$ is even.

**Solution.** (a) Indeed the rows and the columns of the transition probability matrix in this example all add to 1.

(b) We have

$$\sum_{i=1}^{m} \frac{1}{m} p_{ij} = \frac{1}{m}. $$  

Thus, the given probabilities $\pi_j = 1/m$ satisfy the balance equations and must therefore be the steady-state probabilities.

(c) Let $(\pi_1, \ldots, \pi_m)$ be a solution to the balance and normalization equations. Consider a particular $j$ such that $\pi_j \geq \pi_i$ for all $i$, and let $q = \pi_j$. The balance equation for state $j$ yields

$$q = \pi_j = \sum_{i=1}^{m} \pi_i p_{ij} \leq q \sum_{i=1}^{m} p_{ij} = q, $$  

where the last step follows because the transition probability matrix is doubly stochastic. It follows that the above inequality is actually an equality and

$$\sum_{i=1}^{m} \pi_i p_{ij} = \sum_{i=1}^{m} q p_{ij}. $$  

Since $\pi_i \leq q$ for all $i$, we must have $\pi_i p_{ij} = q p_{ij}$ for every $i$. Thus, $\pi_i = q$ for every state $i$ from which a transition to $j$ is possible. By repeating this argument, we see that $\pi_i = q$ for every state $i$ such that there is a positive probability path from $i$ to $j$. Since all states are recurrent and belong to the same class, all states $i$ have this property,
and therefore \( \pi_i \) is the same for all \( i \). Since the \( \pi_i \) add to 1, we obtain \( \pi_1 = 1/m \) for all \( i \).

If \( m \) is even in Example 7.7, the chain is periodic with period 2. Despite this fact, the result we have just established shows that \( \pi_j = 1/m \) is the unique solution to the balance and normalization equations.

**Problem 23.** Queueing. Consider the queueing Example 7.9, but assume that the probabilities of a packet arrival and a packet transmission depend on the state of the queue. In particular, in each period where there are \( i \) packets in the node, one of the following occurs:

(i) one new packet arrives: this happens with a given probability \( b_i \). We assume that \( b_i > 0 \) for \( i < m \) and \( b_m = 0 \).

(ii) one existing packet completes transmission; this happens with a given probability \( d_i > 0 \) if \( i \geq 1 \), and with probability 0 otherwise;

(iii) no new packet arrives and no existing packet completes transmission; this happens with probability \( 1 - b_i - d_i \) if \( i \geq 1 \), and with probability \( 1 - b_i \) if \( i = 0 \).

Calculate the steady-state probabilities of the corresponding Markov chain.

**Solution.** We introduce a Markov chain where the states are 0, 1, \ldots, \( m \), and correspond to the number of packets currently stored at the node. The transition probability graph is given in Fig. 7.23.

![Transition probability graph for Problem 23.](image)

**Figure 7.23:** Transition probability graph for Problem 23.

Similar to Example 7.9, we write down the local balance equations, which take the form

\[
\pi_i b_i = \pi_{i+1} d_{i+1}, \quad i = 0, 1, \ldots, m - 1.
\]

Thus we have \( \pi_{i+1} = \rho_i \pi_i \), where

\[
\rho_i = \frac{b_i}{d_{i+1}}.
\]

Hence \( \pi_i = (\rho_0 \cdots \rho_{i-1}) \pi_0 \) for \( i = 1, \ldots, m \). By using the normalization equation \( 1 = \pi_0 + \pi_1 + \cdots + \pi_m \), we obtain

\[
1 = \pi_0 (1 + \rho_0 + \rho_0 \rho_1 + \cdots + \rho_0 \cdots \rho_{m-1}),
\]

from which

\[
\pi_0 = \frac{1}{1 + \rho_0 + \rho_0 \rho_1 + \cdots + \rho_0 \cdots \rho_{m-1}}.
\]
The remaining steady-state probabilities are

\[ \pi_i = \frac{\rho_0 \cdots \rho_{i-1}}{1 + \rho_0 + \rho_0 \rho_1 + \cdots + \rho_0 \cdots \rho_{m-1}}, \quad i = 1, \ldots, m. \]

**Problem 24.** Dependence of the balance equations. Show that if we add the first \( m - 1 \) balance equations \( \pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \), for \( j = 1, \ldots, m - 1 \), we obtain the last equation \( \pi_m = \sum_{k=1}^{m} \pi_k p_{km} \).

**Solution.** By adding the first \( m - 1 \) balance equations, we obtain

\[
\sum_{j=1}^{m-1} \pi_j = \sum_{j=1}^{m-1} \sum_{k=1}^{m} \pi_k p_{kj} = \sum_{k=1}^{m} \pi_k \sum_{j=1}^{m-1} p_{kj} = \sum_{k=1}^{m} \pi_k (1 - p_{km}) = \pi_m + \sum_{k=1}^{m-1} \pi_k - \sum_{k=1}^{m} \pi_k p_{km}.
\]

This equation is equivalent to the last balance equation \( \pi_m = \sum_{k=1}^{m} \pi_k p_{km} \).

**Problem 25.** Local balance equations. We are given a Markov chain that has a single recurrent class which is aperiodic. Suppose that we have found a solution \( \pi_1, \ldots, \pi_m \) to the following system of local balance and normalization equations:

\[
\pi_i p_{ij} = \pi_j p_{ji}, \quad i, j = 1, \ldots, m,
\]

\[
\sum_{i=1}^{m} \pi_i = 1, \quad i = 1, \ldots, m.
\]

(a) Show that the \( \pi_j \) are the steady-state probabilities.

(b) What is the interpretation of the equations \( \pi_i p_{ij} = \pi_j p_{ji} \) in terms of expected long-term frequencies of transitions between \( i \) and \( j \)?

(c) Construct an example where the local balance equations are not satisfied by the steady-state probabilities.

**Solution.** (a) By adding the local balance equations \( \pi_i p_{ij} = \pi_j p_{ji} \) over \( i \), we obtain

\[
\sum_{i=1}^{m} \pi_i p_{ij} = \sum_{i=1}^{m} \pi_j p_{ji} = \pi_j,
\]

so the \( \pi_j \) also satisfy the balance equations. Therefore, they are equal to the steady-state probabilities.
(b) We know that \( \pi_i p_{ij} \) can be interpreted as the expected long-term frequency of transitions from \( i \) to \( j \), so the local balance equations imply that the expected long-term frequency of any transition is equal to the expected long-term frequency of the reverse transition. (This property is also known as *time reversibility* of the chain.)

(c) We need a minimum of three states for such an example. Let the states be 1, 2, 3, and let \( p_{12} > 0, p_{13} > 0, p_{21} > 0, p_{32} > 0 \), with all other transition probabilities being 0. The chain has a single recurrent aperiodic class. The local balance equations do not hold because the expected frequency of transitions from 1 to 3 is positive, but the expected frequency of reverse transitions is 0.

**Problem 26.** *Sampled Markov chains.* Consider a Markov chain \( X_n \) with transition probabilities \( p_{ij} \), and let \( r_{ij}(n) \) be the \( n \)-step transition probabilities.

(a) Show that for all \( n \geq 1 \) and \( l \geq 1 \), we have

\[
   r_{ij}(n + l) = \sum_{k=1}^{m} r_{ik}(n) r_{kj}(l).
\]

(b) Suppose that there is a single recurrent class, which is aperiodic. We sample the Markov chain every \( l \) transitions, thus generating a process \( Y_n \), where \( Y_n = X_{ln} \). Show that the sampled process can be modeled by a Markov chain with a single aperiodic recurrent class and transition probabilities \( r_{ij}(l) \).

(c) Show that the Markov chain of part (b) has the same steady-state probabilities as the original process.

**Solution.** (a) We condition on \( X_n \) and use the total probability theorem. We have

\[
   r_{ij}(n + l) = P(X_{n+l} = j \mid X_0 = i) \\
   = \sum_{k=1}^{m} P(X_n = k \mid X_0 = i) P(X_{n+l} = j \mid X_n = k, X_0 = i) \\
   = \sum_{k=1}^{m} P(X_n = k \mid X_0 = i) P(X_{n+l} = j \mid X_n = k) \\
   = \sum_{k=1}^{m} r_{ik}(n) r_{kj}(l),
\]

where in the third equality we used the Markov property.

(b) Since \( X_n \) is Markov, once we condition on \( X_{ln} \), the past of the process (the states \( X_k \) for \( k < ln \)) becomes independent of the future (the states \( X_k \) for \( k > ln \)). This implies that given \( Y_n \), the past (the states \( Y_k \) for \( k < n \)) is independent of the future (the states \( Y_k \) for \( k > n \)). Thus, \( Y_n \) has the Markov property. Because of our assumptions on \( X_n \), there is a time \( \bar{n} \) such that

\[
   P(X_n = j \mid X_0 = i) > 0,
\]

for every \( n \geq \bar{n} \), every state \( i \), and every state \( j \) in the single recurrent class \( R \) of the process \( X_n \). This implies that

\[
   P(Y_n = j \mid Y_0 = i) > 0,
\]
for every $n \geq \bar{n}$, every $i$, and every $j \in R$. Therefore, the process $Y_n$ has a single recurrent class, which is aperiodic.

(c) The $n$-step transition probabilities $r_{ij}(n)$ of the process $X_n$ converge to the steady-state probabilities $\pi_j$. The $n$-step transition probabilities of the process $Y_n$ are of the form $r_{ij}(ln)$, and therefore also converge to the same limits $\pi_j$. This establishes that the $\pi_j$ are the steady-state probabilities of the process $Y_n$.

**Problem 27.** Given a Markov chain $X_n$ with a single recurrent class which is aperiodic, consider the Markov chain whose state at time $n$ is $(X_{n-1}, X_n)$. Thus, the state in the new chain can be associated with the last transition in the original chain.

(a) Show that the steady-state probabilities of the new chain are

$$\eta_{ij} = \pi_i p_{ij},$$

where the $\pi_i$ are the steady-state probabilities of the original chain.

(b) Generalize part (a) to the case of the Markov chain $(X_{n-k}, X_{n-k+1}, \ldots, X_n)$, whose state can be associated with the last $k$ transitions of the original chain.

**Solution.** (a) For every state $(i, j)$ of the new Markov chain, we have

$$P\left((X_{n-1}, X_n) = (i, j)\right) = P(X_{n-1} = i) P(X_n = j | X_{n-1} = i) = P(X_{n-1} = i) p_{ij}.$$

Since the Markov chain $X_n$ has a single recurrent class which is aperiodic, $P(X_{n-1} = i)$ converges to the steady-state probability $\pi_i$, for every $i$. It follows that $P\left((X_{n-1}, X_n) = (i, j)\right)$ converges to $\pi_i p_{ij}$, which is therefore the steady-state probability of $(i, j)$.

(b) Using the multiplication rule, we have

$$P\left((X_{n-k}, \ldots, X_n) = (i_0, \ldots, i_k)\right) = P(X_{n-k} = i_0) p_{i_0 i_1} \cdots p_{i_{k-1} i_k}.$$

Therefore, by an argument similar to the one in part (a), the steady-state probability of state $(i_0, \ldots, i_k)$ is equal to $\pi_{i_0} p_{i_0 i_1} \cdots p_{i_{k-1} i_k}$.

**SECTION 7.4. Absorption Probabilities and Expected Time to Absorption**

**Problem 28.** There are $m$ classes offered by a particular department, and each year, the students rank each class from 1 to $m$, in order of difficulty, with rank $m$ being the highest. Unfortunately, the ranking is completely arbitrary. In fact, any given class is equally likely to receive any given rank on a given year (two classes may not receive the same rank). A certain professor chooses to remember only the highest ranking his class has ever gotten.

(a) Find the transition probabilities of the Markov chain that models the ranking that the professor remembers.

(b) Find the recurrent and the transient states.

(c) Find the expected number of years for the professor to achieve the highest ranking given that in the first year he achieved the $i$th ranking.
Problem 29. Consider the Markov chain specified in Fig. 7.24. The steady-state probabilities are known to be:

\[ \pi_1 = \frac{6}{31}, \quad \pi_2 = \frac{9}{31}, \quad \pi_3 = \frac{6}{31}, \quad \pi_4 = \frac{10}{31}. \]

Assume that the process is in state 1 just before the first transition.

(a) What is the probability that the process will be in state 1 just after the sixth transition?

(b) Determine the expected value and variance of the number of transitions up to and including the next transition during which the process returns to state 1.

(c) What is (approximately) the probability that the state of the system resulting from transition 1000 is neither the same as the state resulting from transition 999 nor the same as the state resulting from transition 1001?

Problem 30. Consider the Markov chain specified in Fig. 7.25.

(a) Identify the transient and recurrent states. Also, determine the recurrent classes and indicate which ones, if any, are periodic.

(b) Do there exist steady-state probabilities given that the process starts in state 1? If so, what are they?

(c) Do there exist steady-state probabilities given that the process starts in state 6? If so, what are they?
(d) Assume that the process starts in state 1 but we begin observing it after it reaches steady-state.

(i) Find the probability that the state increases by one during the first transition we observe.

(ii) Find the conditional probability that the process was in state 2 when we started observing it, given that the state increased by one during the first transition that we observed.

(iii) Find the probability that the state increased by one during the first change of state that we observed.

(e) Assume that the process starts in state 4.

(i) For each recurrent class, determine the probability that we eventually reach that class.

(ii) What is the expected number of transitions up to and including the transition at which we reach a recurrent state for the first time?

**Problem 31.* Absorption probabilities.** Consider a Markov chain where each state is either transient or absorbing. Fix an absorbing state $s$. Show that the probabilities $a_i$ of eventually reaching $s$ starting from a state $i$ are the unique solution to the equations

\[
\begin{align*}
a_s &= 1, \\
a_i &= 0, & \text{for all absorbing } i \neq s, \\
a_i &= \sum_{j=1}^{m} p_{ij} a_j, & \text{for all transient } i.
\end{align*}
\]

*Hint:* If there are two solutions, find a system of equations that is satisfied by their difference, and look for its solutions.

*Solution.* The fact that the $a_i$ satisfy these equations was established in the text, using the total probability theorem. To show uniqueness, let $\overline{a}_i$ be another solution, and let $\delta_i = \overline{a}_i - a_i$. Denoting by $\mathcal{A}$ the set of absorbing states and using the fact $\delta_j = 0$ for all $j \in \mathcal{A}$, we have

\[
\delta_i = \sum_{j=1}^{m} p_{ij} \delta_j = \sum_{j \notin \mathcal{A}} p_{ij} \delta_j, & \text{ for all transient } i.
\]

Applying this relation $m$ successive times, we obtain

\[
\delta_i = \sum_{j_1 \notin \mathcal{A}} p_{ij_1} \sum_{j_2 \notin \mathcal{A}} p_{j_1 j_2} \cdots \sum_{j_m \notin \mathcal{A}} p_{j_{m-1} j_m} \cdot \delta_{j_m}.
\]

Hence

\[
|\delta_i| \leq \sum_{j_1 \notin \mathcal{A}} p_{ij_1} \sum_{j_2 \notin \mathcal{A}} p_{j_1 j_2} \cdots \sum_{j_m \notin \mathcal{A}} p_{j_{m-1} j_m} \cdot |\delta_{j_m}|
\]

\[
= \mathbb{P}(X_1 \notin \mathcal{A}, \ldots, X_m \notin \mathcal{A} | X_0 = i) \cdot |\delta_{j_m}|
\]

\[
\leq \mathbb{P}(X_1 \notin \mathcal{A}, \ldots, X_m \notin \mathcal{A} | X_0 = i) \cdot \max_{j \notin \mathcal{A}} |\delta_j|.
\]
The above relation holds for all transient $i$, so we obtain

$$\max_{j \not\in A} |\delta_j| \leq \beta \cdot \max_{j \not\in A} |\delta_j|,$$

where

$$\beta = P(X_1 \not\in A, \ldots, X_m \not\in A | X_0 = i).$$

Note that $\beta < 1$, because there is positive probability that $X_m$ is absorbing, regardless of the initial state. It follows that $\max_{j \not\in A} |\delta_j| = 0$, or $a_i = \overline{a}_i$ for all $i$ that are not absorbing. We also have $a_j = \overline{a}_j$ for all absorbing $j$, so $a_i = \overline{a}_i$ for all $i$.

**Problem 32.** Multiple recurrent classes. Consider a Markov chain that has more than one recurrent class, as well as some transient states. Assume that all the recurrent classes are aperiodic.

(a) For any transient state $i$, let $a_i(k)$ be the probability that starting from $i$ we will reach a state in the $k$th recurrent class. Derive a system of equations whose solution are the $a_i(k)$.

(b) Show that each of the $n$-step transition probabilities $r_{ij}(n)$ converges to a limit, and discuss how these limits can be calculated.

**Solution.** (a) We introduce a new Markov chain that has only transient and absorbing states. The transient states correspond to the transient states of the original, while the absorbing states correspond to the recurrent classes of the original. The transition probabilities $\hat{p}_{ij}$ of the new chain are as follows: if $i$ and $j$ are transient, $\hat{p}_{ij} = p_{ij}$; if $i$ is a transient state and $k$ corresponds to a recurrent class, $\hat{p}_{ik}$ is the sum of the transition probabilities from $i$ to states in the recurrent class in the original Markov chain.

The desired probabilities $a_i(k)$ are the absorption probabilities in the new Markov chain and are given by the corresponding formulas:

$$a_i(k) = \hat{p}_{ik} + \sum_{j: \text{transient}} \hat{p}_{ij}a_j(k), \quad \text{for all transient } i.$$

(b) If $i$ and $j$ are recurrent but belong to different classes, $r_{ij}(n)$ is always 0. If $i$ and $j$ are recurrent but belong to the same class, $r_{ij}(n)$ converges to the steady-state probability of $j$ in a chain consisting of only this particular recurrent class. If $j$ is transient, $r_{ij}(n)$ converges to 0. Finally, if $i$ is transient and $j$ is recurrent, then $r_{ij}(n)$ converges to the product of two probabilities: (1) the probability that starting from $i$ we will reach a state in the recurrent class of $j$, and (2) the steady-state probability of $j$ conditioned on the initial state being in the class of $j$.

**Problem 33.** Mean first passage times. Consider a Markov chain with a single recurrent class, and let $s$ be a fixed recurrent state. Show that the system of equations

$$t_s = 0, \quad t_i = 1 + \sum_{j=1}^{m} p_{ij}t_j, \quad \text{for all } i \neq s,$$

satisfied by the mean first passage times, has a unique solution. **Hint:** If there are two solutions, find a system of equations that is satisfied by their difference, and look for its solutions.
Solution. Let $t_i$ be the mean first passage times. These satisfy the given system of equations. To show uniqueness, let $\tilde{t}_i$ be another solution. Then we have for all $i \neq s$

$$t_i = 1 + \sum_{j \neq s} p_{ij} t_j, \quad \tilde{t}_i = 1 + \sum_{j \neq s} p_{ij} \tilde{t}_j,$$

and by subtraction, we obtain

$$\delta_i = \sum_{j \neq s} p_{ij} \delta_j,$$

where $\delta_i = \tilde{t}_i - t_i$. By applying $m$ successive times this relation, if follows that

$$\delta_i = \sum_{j_1 \neq s} p_{ij_1} \sum_{j_2 \neq s} p_{j_1j_2} \cdots \sum_{j_m \neq s} p_{j_m-1j_m} \cdot \delta_{j_m}.$$

Hence, we have for all $i \neq s$

$$|\delta_i| \leq \sum_{j_1 \neq s} p_{ij_1} \sum_{j_2 \neq s} p_{j_1j_2} \cdots \sum_{j_m \neq s} p_{j_m-1j_m} \cdot \max_j |\delta_j|$$

$$= P(X_1 \neq s, \ldots, X_m \neq s | X_0 = i) \cdot \max_j |\delta_j|.$$

On the other hand, we have $P(X_1 \neq s, \ldots, X_m \neq s | X_0 = i) < 1$. This is because starting from any state there is positive probability that $s$ is reached in $m$ steps. It follows that all the $\delta_i$ must be equal to zero.

Problem 34.* Balance equations and mean recurrence times. Consider a Markov chain with a single recurrent class, and let $s$ be a fixed recurrent state. For any state $i$, let

$$\rho_i = E[\text{Number of visits to } i \text{ between two successive visits to } s],$$

where by convention, $\rho_s = 1$.

(a) Show that for all $i$, we have

$$\rho_i = \sum_{k=1}^{m} \rho_k p_{ki}.$$ 

(b) Show that the numbers

$$\pi_i = \frac{\rho_i}{t_s^*}, \quad i = 1, \ldots, m,$$

sum to 1 and satisfy the balance equations, where $t_s^*$ is the mean recurrence time of $s$ (the expected number of transitions up to the first return to $s$, starting from $s$).

(c) Show that if $\pi_1, \ldots, \pi_m$ are nonnegative, satisfy the balance equations, and sum to 1, then

$$\pi_i = \begin{cases} 
\frac{1}{t_i^*}, & \text{if } i \text{ is recurrent}, \\
0, & \text{if } i \text{ is transient},
\end{cases}$$
where \( t_i^* \) is the mean recurrence time of \( i \).

(d) Show that the distribution of part (b) is the unique probability distribution that satisfies the balance equations.

**Note:** This problem not only provides an alternative proof of the existence and uniqueness of probability distributions that satisfy the balance equations, but also makes an intuitive connection between steady-state probabilities and mean recurrence times. The main idea is to break the process into “cycles,” with a new cycle starting each time that the recurrent state \( s \) is visited. The steady-state probability of \( s \) can be interpreted as the long-term expected frequency of visits to state \( s \), which is inversely proportional to the average time between consecutive visits (the mean recurrence time); cf. part (c). Furthermore, if a state \( i \) is expected to be visited, say, twice as often as some other state \( j \) during a typical cycle, it is plausible that the long-term expected frequency \( \pi_i \) of state \( i \) will be twice as large as \( \pi_j \). Thus, the steady-state probabilities \( \pi_i \) should be proportional to the expected number of visits \( P_i \) during a cycle; cf. part (b).

**Solution.** (a) Consider the Markov chain \( X_n \), initialized with \( X_0 = s \). We claim that for all \( i \)

\[
\rho_i = \sum_{n=1}^{\infty} P(X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = i).
\]

To see this, we first consider the case \( i \neq s \), and let \( I_n \) be the random variable that takes the value 1 if \( X_1 \neq s, \ldots, X_{n-1} \neq s, \) and \( X_n = i \), and the value 0 otherwise. Then, the number of visits to state \( i \) before the next visit to state \( s \) is equal to \( \sum_{n=1}^{\infty} I_n \). Thus,\(^\dagger\)

\[
\rho_i = E\left[ \sum_{n=1}^{\infty} I_n \right] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P(X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = i).
\]

\(^\dagger\) The interchange of the infinite summation and the expectation in the subsequent calculation can be justified by the following argument. We have for any \( k > 0 \),

\[
E\left[ \sum_{n=1}^{\infty} I_n \right] = E\left[ \sum_{n=1}^{k} I_n \right] + E\left[ \sum_{n=k+1}^{\infty} I_n \right] = \sum_{n=1}^{k} E[I_n] + E\left[ \sum_{n=k+1}^{\infty} I_n \right].
\]

Let \( T \) be the first positive time that state \( s \) is visited. Then,

\[
E\left[ \sum_{n=k+1}^{\infty} I_n \right] = \sum_{t=k+2}^{\infty} P(T = t)E\left[ \sum_{n=k+1}^{\infty} I_n \mid T = t \right] \leq \sum_{t=k+2}^{\infty} t P(T = t).
\]

Since the mean recurrence time \( \sum_{t=1}^{\infty} t P(T = t) \) is finite, the limit, as \( k \to \infty \) of \( \sum_{t=k+2}^{\infty} t P(T = t) \) is equal to zero. We take the limit of both sides of the earlier equation, as \( k \to \infty \), to obtain the desired relation

\[
E\left[ \sum_{n=1}^{\infty} I_n \right] = \sum_{n=1}^{\infty} E[I_n].
\]
When $i = s$, the events

$$\{X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = s\},$$

for the different values of $n$, form a partition of the sample space, because they correspond to the different possibilities for the time of the next visit to state $s$. Thus,

$$\sum_{n=1}^{\infty} P(X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = s) = 1 = \rho_s,$$

which completes the verification of our assertion.

We next use the total probability theorem to write for $n \geq 2$,

$$P(X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = i) = \sum_{k \neq s} P(X_1 \neq s, \ldots, X_{n-2} \neq s, X_{n-1} = k)p_{ki}.$$

We thus obtain

$$\rho_i = \sum_{n=1}^{\infty} P(X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = i)$$

$$= p_{si} + \sum_{n=2}^{\infty} P(X_1 \neq s, \ldots, X_{n-1} \neq s, X_n = i)$$

$$= p_{si} + \sum_{n=2}^{\infty} \sum_{k \neq s} P(X_1 \neq s, \ldots, X_{n-2} \neq s, X_{n-1} = k)p_{ki}$$

$$= p_{si} + \sum_{k \neq s} p_{ki} \sum_{n=2}^{\infty} P(X_1 \neq s, \ldots, X_{n-2} \neq s, X_{n-1} = k)$$

$$= \rho_s p_{si} + \sum_{k \neq s} p_{ki} \rho_k$$

$$= \sum_{k=1}^{m} \rho_k p_{ki}.$$  

(b) Dividing both sides of the relation established in part (a) by $t_s^*$, we obtain

$$\pi_i = \sum_{k=1}^{m} \pi_k p_{ki},$$

where $\pi_i = \rho_i / t_s^*$. Thus, the $\pi_i$ solve the balance equations. Furthermore, the $\pi_i$ are nonnegative, and we clearly have $\sum_{i=1}^{m} \pi_i = t_s^*$ or $\sum_{i=1}^{m} \pi_i = 1$. Hence, $(\pi_1, \ldots, \pi_m)$ is a probability distribution.

(c) Consider a probability distribution $(\pi_1, \ldots, \pi_m)$ that satisfies the balance equations. Fix a recurrent state $s$, let $t_s^*$ be the mean recurrence time of $s$, and let $t_i$ be the mean
first passage time from a state \( i \neq s \) to state \( s \). We will show that \( \pi_s t_s^* = 1 \). Indeed, we have

\[
t_s^* = 1 + \sum_{j \neq s} p_{sj} t_j,
\]

\[
t_i = 1 + \sum_{j \neq s} p_{ij} t_j, \quad \text{for all } i \neq s.
\]

Multiplying these equations with \( \pi_s \) and \( \pi_i \), respectively, and adding, we obtain

\[
\pi_s t_s^* + \sum_{i \neq s} \pi_i t_i = 1 + \sum_{i=1}^{m} \pi_i \sum_{j \neq s} p_{ij} t_j.
\]

By using the balance equations, the right-hand side is equal to

\[
1 + \sum_{i=1}^{m} \pi_i \sum_{j \neq s} p_{ij} t_j = 1 + \sum_{j \neq s} t_j \sum_{i=1}^{m} \pi_i p_{ij} = 1 + \sum_{j \neq s} t_j \pi_j.
\]

By combining the last two equations, we obtain \( \pi_s t_s^* = 1 \).

Since the probability distribution \((\pi_1, \ldots, \pi_m)\) satisfies the balance equations, if the initial state \( X_0 \) is chosen according to this distribution, all subsequent states \( X_n \) have the same distribution. If we start at a transient state \( i \), the probability of being at that state at time \( n \) diminishes to 0 as \( n \to \infty \). It follows that we must have \( \pi_i = 0 \).

(d) Part (b) shows that there exists at least one probability distribution that satisfies the balance equations. Part (c) shows that there can be only one such probability distribution.

**Problem 35.* The strong law of large numbers for Markov chains.** Consider a finite-state Markov chain in which all states belong to a single recurrent class which is aperiodic. For a fixed state \( s \), let \( Y_k \) be the time of the \( k \)th visit to state \( s \). Let also \( V_n \) be the number of visits to state \( s \) during the first \( n \) transitions.

(a) Show that \( Y_k/k \) converges with probability 1 to the mean recurrence time \( t_s^* \) of state \( s \).

(b) Show that \( V_n/n \) converges with probability 1 to \( 1/t_s^* \).

(c) Can you relate the limit of \( V_n/n \) to the steady-state probability of state \( s \)?

**Solution.** (a) Let us fix an initial state \( i \), not necessarily the same as \( s \). Thus, the random variables \( Y_{k+1} - Y_k \), for \( k \geq 1 \), correspond to the time between successive visits to state \( s \). Because of the Markov property (the past is independent of the future, given the present), the process “starts fresh” at each revisit to state \( s \) and, therefore, the random variables \( Y_{k+1} - Y_k \) are independent and identically distributed, with mean equal to the mean recurrence time \( t_s^* \). Using the strong law of large numbers, we obtain

\[
\lim_{k \to \infty} \frac{Y_k}{k} = \lim_{k \to \infty} \frac{Y_i}{k} + \lim_{k \to \infty} \left( \frac{Y_2 - Y_1}{k} + \cdots + \frac{Y_k - Y_{k-1}}{k} \right) = 0 + t_s^*,
\]

with probability 1.
(b) Let us fix an element of the sample space (a trajectory of the Markov chain). Let $y_k$ and $v_n$ be the values of the random variables $Y_k$ and $V_n$, respectively. Furthermore, let us assume that the sequence $y_k/k$ converges to $t^*_s$; according to the result of part (a), the set of trajectories with this property has probability 1. Let us consider some $n$ between the time of the $k$th visit to state $s$ and the time just before the next visit to that state:

$$y_k \leq n < y_{k+1}.$$

For every $n$ in this range, we have $v_n = k$, and also

$$\frac{1}{y_{k+1}} < \frac{1}{n} \leq \frac{1}{y_k},$$

from which we obtain

$$\frac{k}{y_{k+1}} \leq \frac{v_n}{n} \leq \frac{k}{y_k}.$$

Note that

$$\lim_{k \to \infty} \frac{k}{y_{k+1}} = \lim_{k \to \infty} \frac{k+1}{k} \cdot \lim_{k \to \infty} \frac{k}{k+1} = \lim_{k \to \infty} \frac{k}{y_k} = \frac{1}{t^*_s}.$$ 

If we now let $n$ go to infinity, the corresponding values of $k$, chosen to satisfy $y_k \leq n < y_{k+1}$ also go to infinity. Therefore, the sequence $v_n/n$ is between two sequences both of which converge to $1/t^*_s$, which implies that the sequence $v_n/n$ converges to $1/t^*_s$ as well. Since this happens for every trajectory in a set of trajectories that has probability equal to 1, we conclude that $V_n/n$ converges to $1/t^*_s$, with probability 1.

(c) We have $1/t^*_s = \pi_s$, as established in Problem 35. This implies the intuitive result that $V_n/n$ converges to $\pi_s$, with probability 1. Note: It is tempting to try to establish the convergence of $V_n/n$ to $\pi_s$ by combining the facts that $V_n/n$ converges [part (b)] together with the fact that $E[V_n]/n$ converges to $\pi_s$ (cf. the long-term expected frequency interpretation of steady-state probabilities in Section 7.3). However, this line of reasoning is not valid. This is because it is generally possible for a sequence $Y_n$ of random variables to converge with probability 1 to a constant, while the expected values converge to a different constant. An example is the following. Let $X$ be uniformly distributed in the unit interval $[0, 1]$. let

$$Y_n = \begin{cases} 0, & \text{if } X \geq 1/n, \\ n, & \text{if } X < 1/n. \end{cases}$$

As long as $X$ is nonzero (which happens with probability 1), the sequence $Y_n$ converges to zero. On the other hand, it can be seen that

$$E[Y_n] = P(X < 1/n) E[Y_n | X < 1/n] = \frac{1}{n} \cdot \frac{n}{2} = \frac{1}{2}, \quad \text{for all } n.$$

SECTION 7.5. Continuous-Time Markov Chains

Problem 36. A facility of $m$ identical machines is sharing a single repairperson. The time to repair a failed machine is exponentially distributed with mean $1/\lambda$. A machine
once operational, fails after a time that is exponentially distributed with mean $1/\mu$. All failure and repair times are independent.

(a) Find the steady-state probability that there is no operational machine.

(b) Find the expected number of operational machines, in steady-state.

**Problem 37.** Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. Penelope arrives at the street corner at a given time. Find her expected waiting time, given that she joins the queue. Assume that the process is in steady-state.

**Problem 38.** There are $m$ users who share a computer system. Each user alternates between "thinking" intervals whose durations are independent exponentially distributed with parameter $\lambda$, and an "active" mode that starts by submitting a service request. The server can only serve one request at a time, and will serve a request completely before serving other requests. The service times of different requests are independent exponentially distributed random variables with parameter $\mu$, and also independent of the thinking times of the users. Construct a Markov chain model and derive the steady-state distribution of the number of pending requests, including the one presently served, if any.

**Problem 39.* Consider a continuous-time Markov chain in which the transition rates $\nu_i$ are the same for all $i$. Assume that the process has a single recurrent class.

(a) Explain why the sequence $Y_n$ of transition times form a Poisson process.

(b) Show that the steady-state probabilities of the Markov chain $X(t)$ are the same as the steady-state probabilities of the embedded Markov chain $X_n$.

**Solution.** (a) Denote by $\nu$ the common value of the transition rates $\nu_i$. The sequence $\{Y_n\}$ is a sequence of independent exponentially distributed time intervals with parameter $\nu$. Therefore they can be associated with the arrival times of a Poisson process with rate $\nu$.

(b) The balance and normalization equations for the continuous-time chain are

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k q_{kj}, \quad j = 1, \ldots, m,$$

$$1 = \sum_{k=1}^{m} \pi_k.$$

By using the relation $q_{jk} = \nu p_{jk}$, and by canceling the common factor $\nu$, these equations are written as

$$\pi_j \sum_{k \neq j} p_{jk} = \sum_{k \neq j} \pi_k p_{kj}, \quad j = 1, \ldots, m,$$

$$1 = \sum_{k=1}^{m} \pi_k.$$
We have $\sum_{k \neq j} p_{jk} = 1 - p_{jj}$, so the first of these two equations is written as

$$\pi_j (1 - p_{jj}) = \sum_{k \neq j} \pi_k p_{kj},$$

or

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, \ldots, m.$$

These are the balance equations for the embedded Markov chain, which have a unique solution since the embedded Markov chain has a single recurrent class, which is aperiodic. Hence the $\pi_j$ are the steady-state probabilities for the embedded Markov chain.