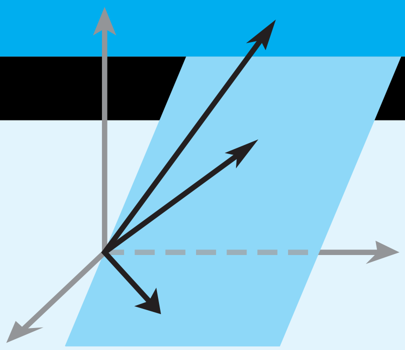


CHAPTER 3

Euclidean Vector Spaces

CHAPTER CONTENTS

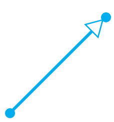
INTRODUCTION Engineers and physicists distinguish between two types of physical quantities—***scalars***, which are quantities that can be described by a numerical value alone, and ***vectors***, which are quantities that require both a number and a direction for their complete physical description. For example, temperature, length, and speed are scalars because they can be fully described by a number that tells "how much"—a temperature of 20°C, a length of 5 cm, or a speed of 75 km/h. In contrast, velocity and force are vectors because they require a number that tells "how much" and a direction that tells "which way"—say, a boat moving at 10 knots in a direction 45° northeast, or a force of 100 lb acting vertically. Although the notions of vectors and scalars that we will study in this text have their origins in physics and engineering, we will be more concerned with using them to build mathematical structures and then applying those structures to such diverse fields as genetics, computer science, economics, telecommunications, and environmental science.

3.1 Vectors in 2-Space, 3-Space, and n-Space

Linear algebra is primarily concerned with two types of mathematical objects, "matrices" and "vectors." In Chapter 1 we discussed the basic properties of matrices, we introduced the idea of viewing n-tuples of real numbers as vectors, and we denoted the set of all such n-tuples as R^n. In this section we will review the basic properties of vectors in two and three dimensions with the goal of extending these properties to vectors in R^n.

Geometric Vectors

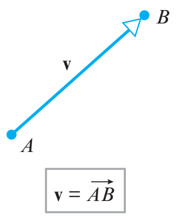
Engineers and physicists represent vectors in two dimensions (also called **2-*space***) or in three dimensions (also called **3-*space***) by arrows. The direction of the arrowhead specifies the ***direction*** of the vector and the ***length*** of the arrow specifies the magnitude. Mathematicians call these ***geometric vectors***. The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point*** (Figure 3.1.1).



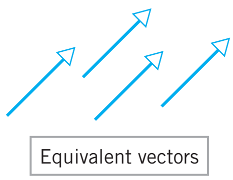
▲ Figure 3.1.1

In this text we will denote vectors in boldface type such as **a**, **b**, **v**, **w**, and **x**, and we will denote scalars in lowercase italic type such as a, k, v, w, and x. When we want to indicate that a vector **v** has initial point A and terminal point B, then, as shown in Figure 3.1.2, we will write

$$\mathbf{v} = \overrightarrow{AB}$$



▲ Figure 3.1.2



▲ Figure 3.1.3

Vectors with the same length and direction, such as those in Figure 3.1.3, are said to be ***equivalent***. Since we want a vector to be determined solely by its length and direction, equivalent vectors are regarded as the same vector even though they may be in different positions. Equivalent vectors are also said to be ***equal***, which we indicate by writing

$$\mathbf{v} = \mathbf{w}$$

The vector whose initial and terminal points coincide has length zero, so we call this the ***zero vector*** and denote it by $\mathbf{0}$. The zero vector has no natural direction, so we will agree that it can be assigned any direction that is convenient for the problem at hand.

Vector Addition

There are a number of important algebraic operations on vectors, all of which have their origin in laws of physics.

Parallelogram Rule for Vector Addition If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram, and the ***sum*** $\mathbf{v} + \mathbf{w}$ is the vector represented by the arrow from the common initial point of $\mathbf{v}$ and $\mathbf{w}$ to the opposite vertex of the parallelogram (Figure 3.1.4*a*).

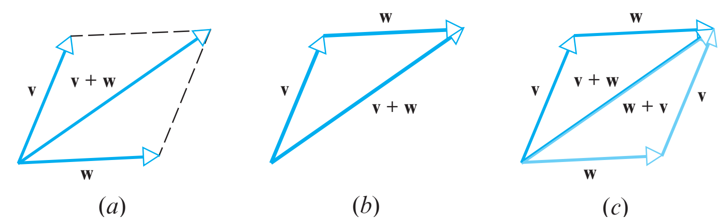
Here is another way to form the sum of two vectors.

Triangle Rule for Vector Addition If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$, then the ***sum*** $\mathbf{v} + \mathbf{w}$ is represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ (Figure 3.1.4*b*).

In Figure 3.1.4*c* we have constructed the sums $\mathbf{v} + \mathbf{w}$ and $\mathbf{w} + \mathbf{v}$ by the triangle rule. This construction makes it evident that

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \tag{1}$$

and that the sum obtained by the triangle rule is the same as the sum obtained by the parallelogram rule.



▶ Figure 3.1.4

Vector addition can also be viewed as a process of translating points.

Vector Addition Viewed as Translation If $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ are positioned so their initial points coincide, then the terminal point of $\mathbf{v} + \mathbf{w}$ can be viewed in two ways:

1. The terminal point of $\mathbf{v} + \mathbf{w}$ is the point that results when the terminal point of $\mathbf{v}$ is translated in the direction of $\mathbf{w}$ by a distance equal to the length of $\mathbf{w}$ (Figure 3.1.5*a*).
2. The terminal point of $\mathbf{v} + \mathbf{w}$ is the point that results when the terminal point of $\mathbf{w}$ is translated in the direction of $\mathbf{v}$ by a distance equal to the length of $\mathbf{v}$ (Figure 3.1.5*b*).

Accordingly, we say that $\mathbf{v} + \mathbf{w}$ is the ***translation of*** $\mathbf{v}$ ***by*** $\mathbf{w}$ or, alternatively, the ***translation of*** $\mathbf{w}$ ***by*** $\mathbf{v}$.

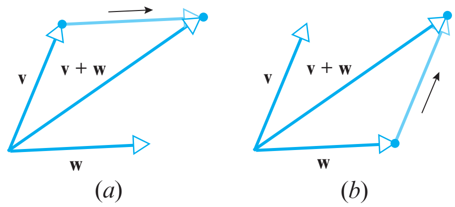


Figure 3.1.5

Vector Subtraction

In ordinary arithmetic we can write $a - b = a + (-b)$, which expresses subtraction in terms of addition. There is an analogous idea in vector arithmetic.

> **Vector Subtraction** The ***negative*** of a vector **v**, denoted by $-\mathbf{v}$, is the vector that has the same length as **v** but is oppositely directed (Figure 3.1.6*a*), and the ***difference*** of **v** from **w**, denoted by $\mathbf{w} - \mathbf{v}$, is taken to be the sum
>
> $$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) \tag{2}$$

The difference of **v** from **w** can be obtained geometrically by the parallelogram method shown in Figure 3.1.6*b*, or more directly by positioning **w** and **v** so their initial points coincide and drawing the vector from the terminal point of **v** to the terminal point of **w** (Figure 3.1.6*c*).

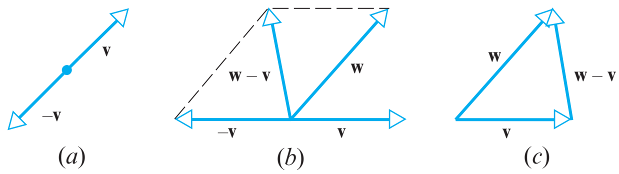


Figure 3.1.6

Scalar Multiplication

Sometimes there is a need to change the length of a vector or change its length and reverse its direction. This is accomplished by a type of multiplication in which vectors are multiplied by scalars. As an example, the product $2\mathbf{v}$ denotes the vector that has the same direction as **v** but twice the length, and the product $-2\mathbf{v}$ denotes the vector that is oppositely directed to **v** and has twice the length. Here is the general result.

> **Scalar Multiplication** If **v** is a nonzero vector in 2-space or 3-space, and if k is a nonzero scalar, then we define the ***scalar product of* v *by* k** to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if k is positive and opposite to that of **v** if k is negative. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then we define $k\mathbf{v}$ to be $\mathbf{0}$.

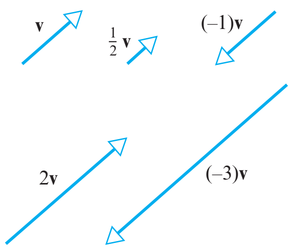


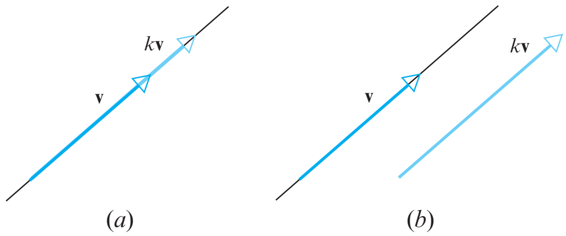
Figure 3.1.7

Figure 3.1.7 shows the geometric relationship between a vector **v** and some of its scalar multiples. In particular, observe that $(-1)\mathbf{v}$ has the same length as **v** but is oppositely directed; therefore,

$$(-1)\mathbf{v} = -\mathbf{v} \tag{3}$$

Parallel and Collinear Vectors

Suppose that **v** and **w** are vectors in 2-space or 3-space with a common initial point. If one of the vectors is a scalar multiple of the other, then the vectors lie on a common line, so it is reasonable to say that they are *collinear* (Figure 3.1.8*a*). However, if we translate one of the vectors, as indicated in Figure 3.1.8*b*, then the vectors are *parallel* but no longer collinear. This creates a linguistic problem because translating a vector does not change it. The only way to resolve this problem is to agree that the terms *parallel* and *collinear* mean the same thing when applied to vectors. Although the vector $\mathbf{0}$ has no clearly defined direction, we will regard it as parallel to all vectors when convenient.



▶ Figure 3.1.8

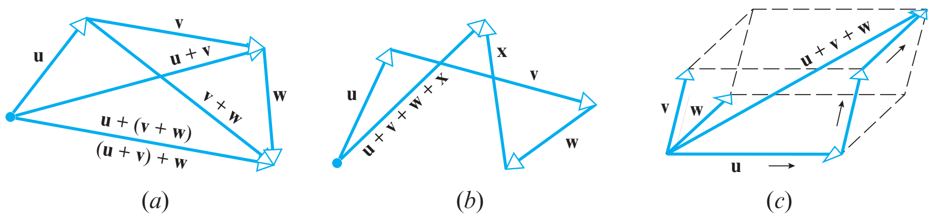
Sums of Three or More Vectors

Vector addition satisfies the ***associative law for addition***, meaning that when we add three vectors, say $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, it does not matter which two we add first; that is,

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

It follows from this that there is no ambiguity in the expression $\mathbf{u} + \mathbf{v} + \mathbf{w}$ because the same result is obtained no matter how the vectors are grouped.

A simple way to construct $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is to place the vectors "tip to tail" in succession and then draw the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{w}$ (Figure 3.1.9*a*). The tip-to-tail method also works for four or more vectors (Figure 3.1.9*b*). The tip-to-tail method makes it evident that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in 3-space with a *common initial point*, then $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is the diagonal of the parallelepiped that has the three vectors as adjacent sides (Figure 3.1.9*c*).



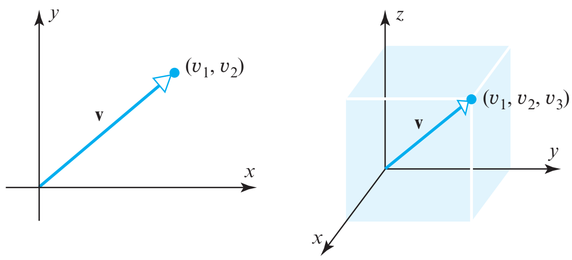
▶ Figure 3.1.9

Vectors in Coordinate Systems

Up until now we have discussed vectors without reference to a coordinate system. However, as we will soon see, computations with vectors are much simpler to perform if a coordinate system is present to work with.

If a vector $\mathbf{v}$ in 2-space or 3-space is positioned with its initial point at the origin of a rectangular coordinate system, then the vector is completely determined by the coordinates of its terminal point (Figure 3.1.10). We call these coordinates the ***components*** of $\mathbf{v}$ relative to the coordinate system. We will write $\mathbf{v} = (v_1, v_2)$ to denote a vector $\mathbf{v}$ in 2-space with components (v_1, v_2), and $\mathbf{v} = (v_1, v_2, v_3)$ to denote a vector $\mathbf{v}$ in 3-space with components (v_1, v_2, v_3).

The component forms of the zero vector are $\mathbf{0} = (0, 0)$ in 2-space and $\mathbf{0} = (0, 0, 0)$ in 3-space.



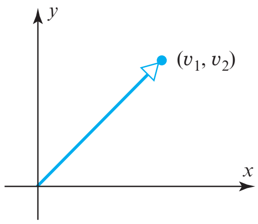
▶ Figure 3.1.10

It should be evident geometrically that two vectors in 2-space or 3-space are equivalent if and only if they have the same terminal point when their initial points are at the origin. Algebraically, this means that two vectors are equivalent if and only if their corresponding components are equal. Thus, for example, the vectors

$$\mathbf{v} = (v_1, v_2, v_3) \quad \text{and} \quad \mathbf{w} = (w_1, w_2, w_3)$$

in 3-space are equivalent if and only if

$$v_1 = w_1, \quad v_2 = w_2, \quad v_3 = w_3$$



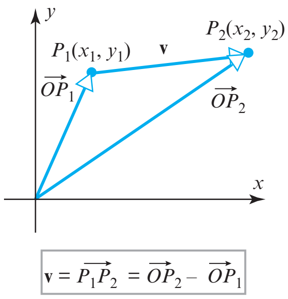
▲ Figure 3.1.11 The ordered pair (v_1, v_2) can represent a point or a vector.

Remark It may have occurred to you that an ordered pair (v_1, v_2) can represent either a vector with *components* v_1 and v_2 or a point with *coordinates* v_1 and v_2 (and similarly for ordered triples). Both are valid geometric interpretations, so the appropriate choice will depend on the geometric viewpoint that we want to emphasize (Figure 3.1.11).

Vectors Whose Initial Point Is Not at the Origin

It is sometimes necessary to consider vectors whose initial points are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1) \tag{4}$$



▲ Figure 3.1.12

That is, the components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point. For example, in Figure 3.1.12 the vector $\overrightarrow{P_1P_2}$ is the difference of vectors $\overrightarrow{OP_2}$ and $\overrightarrow{OP_1}$, so

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$$

As you might expect, the components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \tag{5}$$

▶ EXAMPLE 1 Finding the Components of a Vector

The components of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$ are

$$\mathbf{v} = (7 - 2, 5 - (-1), (-8) - 4) = (5, 6, -12) \quad ◀$$

n-Space

The idea of using ordered pairs and triples of real numbers to represent points in two-dimensional space and three-dimensional space was well known in the eighteenth and nineteenth centuries. By the dawn of the twentieth century, mathematicians and physicists were exploring the use of "higher dimensional" spaces in mathematics and physics. Today, even the layman is familiar with the notion of time as a fourth dimension, an idea used by Albert Einstein in developing the general theory of relativity. Today, physicists working in the field of "string theory" commonly use 11-dimensional space in their quest for a unified theory that will explain how the fundamental forces of nature work. Much of the remaining work in this section is concerned with extending the notion of space to n dimensions.

To explore these ideas further, we start with some terminology and notation. The set of all real numbers can be viewed geometrically as a line. It is called the ***real line*** and is denoted by R or R^1. The superscript reinforces the intuitive idea that a line is one-dimensional. The set of all ordered pairs of real numbers (called **2-*tuples***) and the set of all ordered triples of real numbers (called **3-*tuples***) are denoted by R^2 and R^3, respectively.

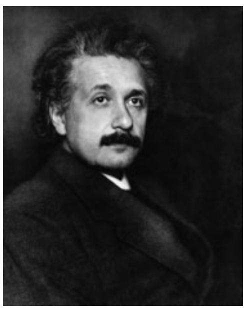
The superscript reinforces the idea that the ordered pairs correspond to points in the plane (two-dimensional) and ordered triples to points in space (three-dimensional). The following definition extends this idea.

DEFINITION 1 If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by R^n.

Remark You can think of the numbers in an n-tuple $(v_1, v_2, \ldots, v_n)$ as either the coordinates of a *generalized point* or the components of a *generalized vector*, depending on the geometric image you want to bring to mind—the choice makes no difference mathematically, since it is the algebraic properties of n-tuples that are of concern.

Here are some typical applications that lead to n-tuples.

- **Experimental Data**—A scientist performs an experiment and makes n numerical measurements each time the experiment is performed. The result of each experiment can be regarded as a vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in R^n in which $y_1, y_2, \ldots, y_n$ are the measured values.
- **Storage and Warehousing**—A national trucking company has 15 depots for storing and servicing its trucks. At each point in time the distribution of trucks in the service depots can be described by a 15-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_{15})$ in which x_1 is the number of trucks in the first depot, x_2 is the number in the second depot, and so forth.
- **Electrical Circuits**—A certain kind of processing chip is designed to receive four input voltages and produce three output voltages in response. The input voltages can be regarded as vectors in R^4 and the output voltages as vectors in R^3. Thus, the chip can be viewed as a device that transforms an input vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ in R^4 into an output vector $\mathbf{w} = (w_1, w_2, w_3)$ in R^3.
- **Graphical Images**—One way in which color images are created on computer screens is by assigning each pixel (an addressable point on the screen) three numbers that describe the ***hue***, ***saturation***, and ***brightness*** of the pixel. Thus, a complete color image can be viewed as a set of 5-tuples of the form $\mathbf{v} = (x, y, h, s, b)$ in which x and y are the screen coordinates of a pixel and h, s, and b are its hue, saturation, and brightness.
- **Economics**—One approach to economic analysis is to divide an economy into sectors (manufacturing, services, utilities, and so forth) and measure the output of each sector by a dollar value. Thus, in an economy with 10 sectors the economic output of the entire economy can be represented by a 10-tuple $\mathbf{s} = (s_1, s_2, \ldots, s_{10})$ in which the numbers $s_1, s_2, \ldots, s_{10}$ are the outputs of the individual sectors.



Albert Einstein (1879–1955)

Historical Note The German-born physicist Albert Einstein immigrated to the United States in 1935, where he settled at Princeton University. Einstein spent the last three decades of his life working unsuccessfully at producing a *unified field theory* that would establish an underlying link between the forces of gravity and electromagnetism. Recently, physicists have made progress on the problem using a framework known as *string theory*. In this theory the smallest, indivisible components of the Universe are not particles but loops that behave like vibrating strings. Whereas Einstein's space-time universe was four-dimensional, strings reside in an 11-dimensional world that is the focus of current research.

- **Mechanical Systems**—Suppose that six particles move along the same coordinate line so that at time t their coordinates are $x_1, x_2, \dots, x_6$ and their velocities are $v_1, v_2, \dots, v_6$, respectively. This information can be represented by the vector

$$\mathbf{v} = (x_1, x_2, x_3, x_4, x_5, x_6, v_1, v_2, v_3, v_4, v_5, v_6, t)$$

in R^{13}. This vector is called the ***state*** of the particle system at time t.

Operations on Vectors in R^n

Our next goal is to define useful operations on vectors in R^n. These operations will all be natural extensions of the familiar operations on vectors in R^2 and R^3. We will denote a vector $\mathbf{v}$ in R^n using the notation

$$\mathbf{v} = (v_1, v_2, \dots, v_n)$$

and we will call $\mathbf{0} = (0, 0, \dots, 0)$ the ***zero vector***.

We noted earlier that in R^2 and R^3 two vectors are equivalent (equal) if and only if their corresponding components are the same. Thus, we make the following definition.

DEFINITION 2 Vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ in R^n are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \dots, \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

▶ EXAMPLE 2 Equality of Vectors

$$(a, b, c, d) = (1, -4, 2, 7)$$

if and only if $a = 1$, $b = -4$, $c = 2$, and $d = 7$. ◀

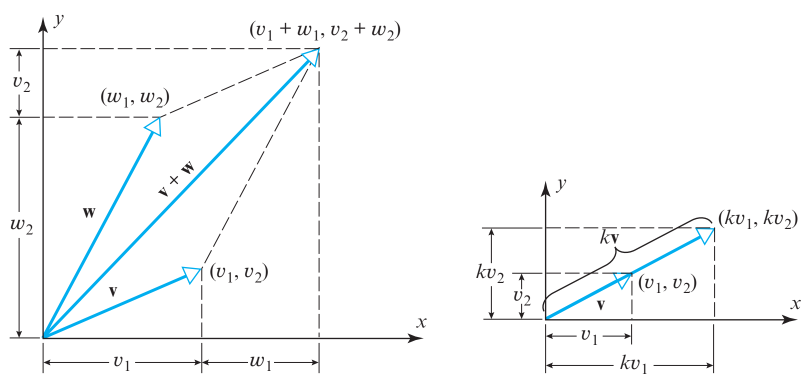
Our next objective is to define the operations of addition, subtraction, and scalar multiplication for vectors in R^n. To motivate these ideas, we will consider how these operations can be performed on vectors in R^2 using components. By studying Figure 3.1.13 you should be able to deduce that if $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, then

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2) \tag{6}$$

$$k\mathbf{v} = (kv_1, kv_2) \tag{7}$$

In particular, it follows from (7) that

$$-\mathbf{v} = (-1)\mathbf{v} = (-v_1, -v_2) \tag{8}$$



▶ Figure 3.1.13

and hence that

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2) \tag{9}$$

Motivated by Formulas (6)–(9), we make the following definition.

DEFINITION 3 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \tag{10}$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \tag{11}$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \tag{12}$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n) \tag{13}$$

▶ EXAMPLE 3 Algebraic Operations Using Components

In words, vectors are added (or subtracted) by adding (or subtracting) their corresponding components, and a vector is multiplied by a scalar by multiplying each component by that scalar.

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4)$$

$$-\mathbf{w} = (-4, -2, -1), \qquad \mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)$$ ◀

The following theorem summarizes the most important properties of vector operations.

THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*f*) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(*g*) $k(m\mathbf{u}) = (km)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

We will prove part (*b*) and leave some of the other proofs as exercises.

Proof (b) Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Then

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \big((u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)\big) + (w_1, w_2, \ldots, w_n) \\
&= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) + (w_1, w_2, \ldots, w_n) && \text{[Vector addition]} \\
&= \big((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n\big) && \text{[Vector addition]} \\
&= \big(u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)\big) && \text{[Regroup]} \\
&= (u_1, u_2, \ldots, u_n) + (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) && \text{[Vector addition]} \\
&= \mathbf{u} + (\mathbf{v} + \mathbf{w})
\end{aligned}$$ ◀

The following additional properties of vectors in R^n can be deduced easily by expressing the vectors in terms of components (verify).

THEOREM 3.1.2 *If* $\mathbf{v}$ *is a vector in* R^n *and* k *is a scalar, then:*

(a) $0\mathbf{v} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{v} = -\mathbf{v}$

Calculating Without Components

One of the powerful consequences of Theorems 3.1.1 and 3.1.2 is that they allow calculations to be performed without expressing the vectors in terms of components. For example, suppose that $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ are vectors in R^n, and we want to solve the vector equation $\mathbf{x} + \mathbf{a} = \mathbf{b}$ for the vector $\mathbf{x}$ without using components. We could proceed as follows:

$$\mathbf{x} + \mathbf{a} = \mathbf{b} \qquad \textbf{[Given]}$$

$$(\mathbf{x} + \mathbf{a}) + (-\mathbf{a}) = \mathbf{b} + (-\mathbf{a}) \qquad \textbf{[Add the negative of a to both sides]}$$

$$\mathbf{x} + (\mathbf{a} + (-\mathbf{a})) = \mathbf{b} - \mathbf{a} \qquad \textbf{[Part (b) of Theorem 3.1.1]}$$

$$\mathbf{x} + \mathbf{0} = \mathbf{b} - \mathbf{a} \qquad \textbf{[Part (d) of Theorem 3.1.1]}$$

$$\mathbf{x} = \mathbf{b} - \mathbf{a} \qquad \textbf{[Part (c) of Theorem 3.1.1]}$$

While this method is obviously more cumbersome than computing with components in R^n, it will become important later in the text where we will encounter more general kinds of vectors.

Linear Combinations

Addition, subtraction, and scalar multiplication are frequently used in combination to form new vectors. For example, if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in R^n, then the vectors

$$\mathbf{u} = 2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 \quad \text{and} \quad \mathbf{w} = 7\mathbf{v}_1 - 6\mathbf{v}_2 + 8\mathbf{v}_3$$

are formed in this way. In general, we make the following definition.

Note that this definition of a linear combination is consistent with that given in the context of matrices (see Definition 6 in Section 1.3).

DEFINITION 4 If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{14}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination. In the case where $r = 1$, Formula (14) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

Alternative Notations for Vectors

Up to now we have been writing vectors in R^n using the notation

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) \tag{15}$$

We call this the ***comma-delimited*** form. However, since a vector in R^n is just a list of its n components in a specific order, any notation that displays those components in the correct order is a valid way of representing the vector. For example, the vector in (15) can be written as

$$\mathbf{v} = [v_1 \quad v_2 \quad \cdots \quad v_n] \tag{16}$$

which is called ***row-vector*** form, or as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{17}$$

which is called ***column-vector*** form. The choice of notation is often a matter of taste or convenience, but sometimes the nature of a problem will suggest a preferred notation. Notations (15), (16), and (17) will all be used at various places in this text.

Application of Linear Combinations to Color Models

Colors on computer monitors are commonly based on what is called the **RGB *color model***. Colors in this system are created by adding together percentages of the primary colors red (R), green (G), and blue (B). One way to do this is to identify the primary colors with the vectors

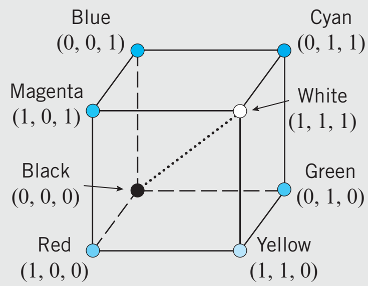
$$\mathbf{r} = (1, 0, 0) \quad \text{(pure red)},$$
$$\mathbf{g} = (0, 1, 0) \quad \text{(pure green)},$$
$$\mathbf{b} = (0, 0, 1) \quad \text{(pure blue)}$$

in R^3 and to create all other colors by forming linear combinations of **r**, **g**, and **b** using coefficients between 0 and 1, inclusive; these coefficients represent the percentage of each pure color in the mix.

The set of all such color vectors is called **RGB *space*** or the **RGB *color cube*** (Figure 3.1.14). Thus, each color vector **c** in this cube is expressible as a linear combination of the form

$$\begin{aligned}\mathbf{c} &= k_1\mathbf{r} + k_2\mathbf{g} + k_3\mathbf{b} \\ &= k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) \\ &= (k_1, k_2, k_3)\end{aligned}$$

where $0 \le k_i \le 1$. As indicated in the figure, the corners of the cube represent the pure primary colors together with the colors black, white, magenta, cyan, and yellow. The vectors along the diagonal running from black to white correspond to shades of gray.

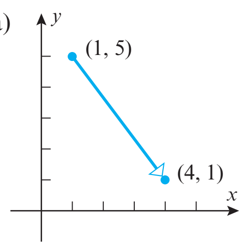


▶ Figure 3.1.14

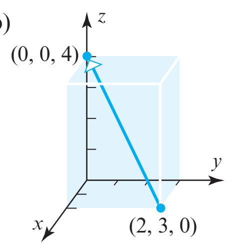
Exercise Set 3.1

▶ In Exercises 1–2, find the components of the vector. ◀

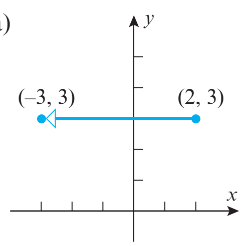
1. (a)



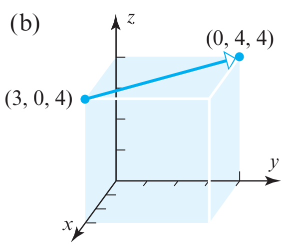
(b)



2. (a)



(b)



▶ In Exercises 3–4, find the components of the vector $\overrightarrow{P_1P_2}$. ◀

3. (a) $P_1(3, 5)$, $P_2(2, 8)$ (b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$

4. (a) $P_1(-6, 2)$, $P_2(-4, -1)$ (b) $P_1(0, 0, 0)$, $P_2(-1, 6, 1)$

5. (a) Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1, 2)$ and whose initial point is $A(1, 1)$.

(b) Find the initial point of the vector that is equivalent to $\mathbf{u} = (1, 1, 3)$ and whose terminal point is $B(-1, -1, 2)$.

6. (a) Find the initial point of the vector that is equivalent to $\mathbf{u} = (1, 2)$ and whose terminal point is $B(2, 0)$.

(b) Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1, 1, 3)$ and whose initial point is $A(0, 2, 0)$.

7. Find an initial point P of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with terminal point $Q(3, 0, -5)$ and such that

(a) **u** has the same direction as $\mathbf{v} = (4, -2, -1)$.

(b) **u** is oppositely directed to $\mathbf{v} = (4, -2, -1)$.

8. Find a terminal point Q of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with initial point $P(-1, 3, -5)$ and such that

(a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (6, 7, -3)$.

(b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (6, 7, -3)$.

9. Let $\mathbf{u} = (4, -1)$, $\mathbf{v} = (0, 5)$, and $\mathbf{w} = (-3, -3)$. Find the components of

(a) $\mathbf{u} + \mathbf{w}$ (b) $\mathbf{v} - 3\mathbf{u}$

(c) $2(\mathbf{u} - 5\mathbf{w})$ (d) $3\mathbf{v} - 2(\mathbf{u} + 2\mathbf{w})$

10. Let $\mathbf{u} = (-3, 1, 2)$, $\mathbf{v} = (4, 0, -8)$, and $\mathbf{w} = (6, -1, -4)$. Find the components of

(a) $\mathbf{v} - \mathbf{w}$ (b) $6\mathbf{u} + 2\mathbf{v}$

(c) $-3(\mathbf{v} - 8\mathbf{w})$ (d) $(2\mathbf{u} - 7\mathbf{w}) - (8\mathbf{v} + \mathbf{u})$

11. Let $\mathbf{u} = (-3, 2, 1, 0)$, $\mathbf{v} = (4, 7, -3, 2)$, and $\mathbf{w} = (5, -2, 8, 1)$. Find the components of

(a) $\mathbf{v} - \mathbf{w}$ (b) $-\mathbf{u} + (\mathbf{v} - 4\mathbf{w})$

(c) $6(\mathbf{u} - 3\mathbf{v})$ (d) $(6\mathbf{v} - \mathbf{w}) - (4\mathbf{u} + \mathbf{v})$

12. Let $\mathbf{u} = (1, 2, -3, 5, 0)$, $\mathbf{v} = (0, 4, -1, 1, 2)$, and $\mathbf{w} = (7, 1, -4, -2, 3)$. Find the components of

(a) $\mathbf{v} + \mathbf{w}$ (b) $3(2\mathbf{u} - \mathbf{v})$

(c) $(3\mathbf{u} - \mathbf{v}) - (2\mathbf{u} + 4\mathbf{w})$ (d) $\frac{1}{2}(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v}$

13. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the vectors in Exercise 11. Find the components of the vector $\mathbf{x}$ that satisfies the equation $3\mathbf{u} + \mathbf{v} - 2\mathbf{w} = 3\mathbf{x} + 2\mathbf{w}$.

14. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the vectors in Exercise 12. Find the components of the vector $\mathbf{x}$ that satisfies the equation $2\mathbf{u} - \mathbf{v} + \mathbf{x} = 7\mathbf{x} + \mathbf{w}$.

15. Which of the following vectors in R^6, if any, are parallel to $\mathbf{u} = (-2, 1, 0, 3, 5, 1)$?

(a) $(4, 2, 0, 6, 10, 2)$

(b) $(4, -2, 0, -6, -10, -2)$

(c) $(0, 0, 0, 0, 0, 0)$

16. For what value(s) of t, if any, is the given vector parallel to $\mathbf{u} = (4, -1)$?

(a) $(8t, -2)$ (b) $(8t, 2t)$ (c) $(1, t^2)$

17. Let $\mathbf{u} = (1, -1, 3, 5)$ and $\mathbf{v} = (2, 1, 0, -3)$. Find scalars a and b so that $a\mathbf{u} + b\mathbf{v} = (1, -4, 9, 18)$.

18. Let $\mathbf{u} = (2, 1, 0, 1, -1)$ and $\mathbf{v} = (-2, 3, 1, 0, 2)$. Find scalars a and b so that $a\mathbf{u} + b\mathbf{v} = (-8, 8, 3, -1, 7)$.

▶ In Exercises 19–20, find scalars c_1, c_2, and c_3 for which the equation is satisfied. ◀

19. $c_1(1, -1, 0) + c_2(3, 2, 1) + c_3(0, 1, 4) = (-1, 1, 19)$

20. $c_1(-1, 0, 2) + c_2(2, 2, -2) + c_3(1, -2, 1) = (-6, 12, 4)$

21. Show that there do not exist scalars c_1, c_2, and c_3 such that

$$c_1(-2, 9, 6) + c_2(-3, 2, 1) + c_3(1, 7, 5) = (0, 5, 4)$$

22. Show that there do not exist scalars c_1, c_2, and c_3 such that

$$c_1(1, 0, 1, 0) + c_2(1, 0, -2, 1) + c_3(2, 0, 1, 2) = (1, -2, 2, 3)$$

23. Let P be the point $(2, 3, -2)$ and Q the point $(7, -4, 1)$.

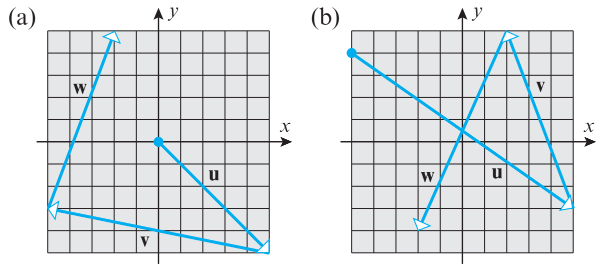
(a) Find the midpoint of the line segment connecting the points P and Q.

(b) Find the point on the line segment connecting the points P and Q that is $\frac{3}{4}$ of the way from P to Q.

24. In relation to the points P_1 and P_2 in Figure 3.1.12, what can you say about the terminal point of the following vector if its initial point is at the origin?

$$\mathbf{u} = \overrightarrow{OP_1} + \tfrac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})$$

25. In each part, find the components of the vector $\mathbf{u} + \mathbf{v} + \mathbf{w}$.



26. Referring to the vectors pictured in Exercise 25, find the components of the vector $\mathbf{u} - \mathbf{v} + \mathbf{w}$.

27. Let P be the point $(1, 3, 7)$. If the point $(4, 0, -6)$ is the midpoint of the line segment connecting P and Q, what is Q?

28. If the sum of three vectors in R^3 is zero, must they lie in the same plane? Explain.

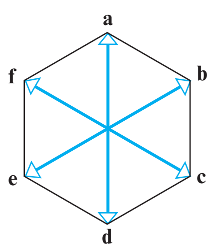
29. Consider the regular hexagon shown in the accompanying figure.

(a) What is the sum of the six radial vectors that run from the center to the vertices?

(b) How is the sum affected if each radial vector is multiplied by $\frac{1}{2}$?

(c) What is the sum of the five radial vectors that remain if $\mathbf{a}$ is removed?

(d) Discuss some variations and generalizations of the result in part (c).



◀ Figure Ex-29

30. What is the sum of all radial vectors of a regular n-sided polygon? (See Exercise 29.)

Working with Proofs

31. Prove parts (*a*), (*c*), and (*d*) of Theorem 3.1.1.

32. Prove parts (*e*)–(*h*) of Theorem 3.1.1.

33. Prove parts (*a*)–(*c*) of Theorem 3.1.2.

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two equivalent vectors must have the same initial point.

(b) The vectors (a, b) and $(a, b, 0)$ are equivalent.

(c) If k is a scalar and $\mathbf{v}$ is a vector, then $\mathbf{v}$ and $k\mathbf{v}$ are parallel if and only if $k \geq 0$.

(d) The vectors $\mathbf{v} + (\mathbf{u} + \mathbf{w})$ and $(\mathbf{w} + \mathbf{v}) + \mathbf{u}$ are the same.

(e) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(f) If a and b are scalars such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors.

(g) Collinear vectors with the same length are equal.

(h) If $(a, b, c) + (x, y, z) = (x, y, z)$, then (a, b, c) must be the zero vector.

(i) If k and m are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors, then

$$(k + m)(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + m\mathbf{v}$$

(j) If the vectors $\mathbf{v}$ and $\mathbf{w}$ are given, then the vector equation

$$3(2\mathbf{v} - \mathbf{x}) = 5\mathbf{x} - 4\mathbf{w} + \mathbf{v}$$

can be solved for $\mathbf{x}$.

(k) The linear combinations $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ and $b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ can only be equal if $a_1 = b_1$ and $a_2 = b_2$.

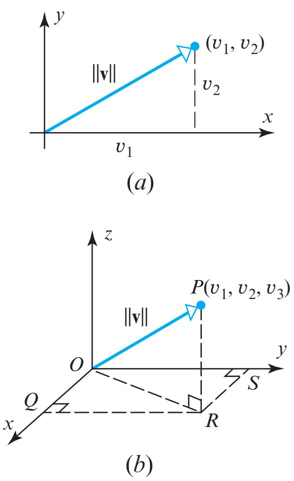
3.2 Norm, Dot Product, and Distance in R^n

In this section we will be concerned with the notions of length and distance as they relate to vectors. We will first discuss these ideas in R^2 and R^3 and then extend them algebraically to R^n.

Norm of a Vector

In this text we will denote the length of a vector $\mathbf{v}$ by the symbol $\|\mathbf{v}\|$, which is read as the *norm* of $\mathbf{v}$, the *length* of $\mathbf{v}$, or the *magnitude* of $\mathbf{v}$ (the term "norm" being a common mathematical synonym for length). As suggested in Figure 3.2.1*a*, it follows from the Theorem of Pythagoras that the norm of a vector (v_1, v_2) in R^2 is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \tag{1}$$



▲ Figure 3.2.1

Similarly, for a vector (v_1, v_2, v_3) in R^3, it follows from Figure 3.2.1*b* and two applications of the Theorem of Pythagoras that

$$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (QR)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2$$

and hence that

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{2}$$

Motivated by the pattern of Formulas (1) and (2), we make the following definition.

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

EXAMPLE 1 Calculating Norms

It follows from Formula (2) that the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and it follows from Formula (3) that the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$

Our first theorem in this section will generalize to R^n the following three familiar facts about vectors in R^2 and R^3:

- Distances are nonnegative.
- The zero vector is the only vector of length zero.
- Multiplying a vector by a scalar multiplies its length by the absolute value of that scalar.

It is important to recognize that just because these results hold in R^2 and R^3 does not guarantee that they hold in R^n—their validity in R^n must be *proved* using algebraic properties of n-tuples.

THEOREM 3.2.1 *If* $\mathbf{v}$ *is a vector in* R^n, *and if* k *is any scalar, then*:

(*a*) $\|\mathbf{v}\| \geq 0$

(*b*) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(*c*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

We will prove part (*c*) and leave (*a*) and (*b*) as exercises.

Proof (c) If $\mathbf{v} = (v_1, v_2, \dots, v_n)$, then $k\mathbf{v} = (kv_1, kv_2, \dots, kv_n)$, so

$$\begin{aligned}\|k\mathbf{v}\| &= \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2} \\ &= \sqrt{(k^2)(v_1^2 + v_2^2 + \cdots + v_n^2)} \\ &= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= |k|\|\mathbf{v}\|\end{aligned}$$

Unit Vectors

A vector of norm 1 is called a ***unit vector***. Such vectors are useful for specifying a direction when length is not relevant to the problem at hand. You can obtain a unit vector in a desired direction by choosing any *nonzero* vector $\mathbf{v}$ in that direction and multiplying $\mathbf{v}$ by the reciprocal of its length. For example, if $\mathbf{v}$ is a vector of length 2 in R^2 or R^3, then $\frac{1}{2}\mathbf{v}$ is a unit vector in the same direction as $\mathbf{v}$. More generally, if $\mathbf{v}$ is any nonzero vector in R^n, then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \tag{4}$$

defines a unit vector that is in the same direction as $\mathbf{v}$. We can confirm that (4) is a unit vector by applying part (*c*) of Theorem 3.2.1 with $k = 1/\|\mathbf{v}\|$ to obtain

$$\|\mathbf{u}\| = \|k\mathbf{v}\| = |k|\|\mathbf{v}\| = k\|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

WARNING Sometimes you will see Formula (4) expressed as

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

This is just a more compact way of writing that formula and is *not* intended to convey that $\mathbf{v}$ is being divided by $\|\mathbf{v}\|$.

The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called ***normalizing*** **v**.

▶ EXAMPLE 2 Normalizing a Vector

Find the unit vector **u** that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector **v** has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from (4)

$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

As a check, you may want to confirm that $\|\mathbf{u}\| = 1$. ◀

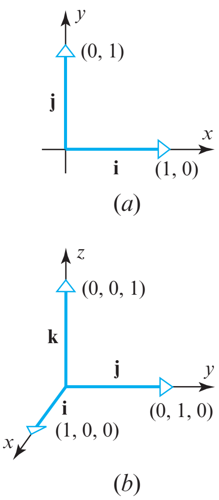
The Standard Unit Vectors

When a rectangular coordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinate axes are called the ***standard unit vectors***. In R^2 these vectors are denoted by

$$\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1)$$

and in R^3 by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1)$$



▲ Figure 3.2.2

(Figure 3.2.2). Every vector $\mathbf{v} = (v_1, v_2)$ in R^2 and every vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 can be expressed as a linear combination of standard unit vectors by writing

$$\mathbf{v} = (v_1, v_2) = v_1(1, 0) + v_2(0, 1) = v_1\mathbf{i} + v_2\mathbf{j} \tag{5}$$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \tag{6}$$

Moreover, we can generalize these formulas to R^n by defining the ***standard unit vectors in*** $\boldsymbol{R^n}$ to be

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1) \tag{7}$$

in which case every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n \tag{8}$$

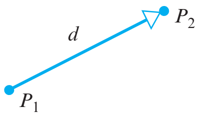
▶ EXAMPLE 3 Linear Combinations of Standard Unit Vectors

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$
$$(7, 3, -4, 5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4$$ ◀

Distance in R^n

If P_1 and P_2 are points in R^2 or R^3, then the length of the vector $\overrightarrow{P_1P_2}$ is equal to the distance d between the two points (Figure 3.2.3). Specifically, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in R^2, then Formula (4) of Section 3.1 implies that

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{9}$$



$d = \|\overrightarrow{P_1P_2}\|$

▲ Figure 3.2.3

We noted in the previous section that n-tuples can be viewed either as vectors or points in R^n. In Definition 2 we chose to describe them as points, as that seemed the more natural interpretation.

This is the familiar distance formula from analytic geometry. Similarly, the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space is

$$d(\mathbf{u}, \mathbf{v}) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{10}$$

Motivated by Formulas (9) and (10), we make the following definition.

DEFINITION 2 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

▶ EXAMPLE 4 Calculating Distance in R^n

If

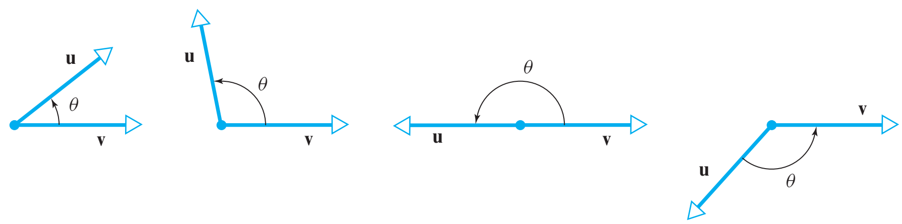
$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$ ◀

Dot Product

Our next objective is to define a useful multiplication operation on vectors in R^2 and R^3 and then extend that operation to R^n. To do this we will first need to define exactly what we mean by the "angle" between two vectors in R^2 or R^3. For this purpose, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^2 or R^3 that have been positioned so that their initial points coincide. We define the ***angle between* u *and* v** to be the angle θ determined by $\mathbf{u}$ and $\mathbf{v}$ that satisfies the inequalities $0 \leq \theta \leq \pi$ (Figure 3.2.4).



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$.

▶ Figure 3.2.4

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{12}$$

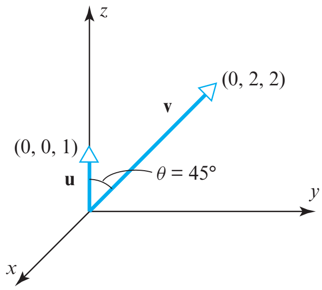
If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{13}$$

Since $0 \leq \theta \leq \pi$, it follows from Formula (13) and properties of the cosine function studied in trigonometry that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$.



▲ Figure 3.2.5

EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

Solution The lengths of the vectors are

$$\|\mathbf{u}\| = 1 \quad \text{and} \quad \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$$

and the cosine of the angle θ between them is

$$\cos(45^\circ) = 1/\sqrt{2}$$

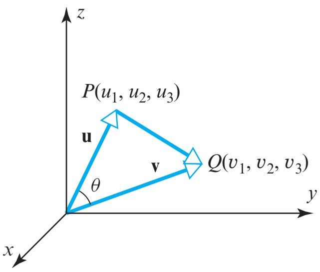
Thus, it follows from Formula (12) that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = (1)(2\sqrt{2})(1/\sqrt{2}) = 2$$ ◀

Component Form of the Dot Product

For computational purposes it is desirable to have a formula that expresses the dot product of two vectors in terms of components. We will derive such a formula for vectors in 3-space; the derivation for vectors in 2-space is similar.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two nonzero vectors. If, as shown in Figure 3.2.6, θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the law of cosines yields



▲ Figure 3.2.6

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{14}$$

Since $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$, we can rewrite (14) as

$$\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

or

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

Substituting

$$\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + u_3^2, \qquad \|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

and

$$\|\mathbf{v} - \mathbf{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2$$

we obtain, after simplifying,

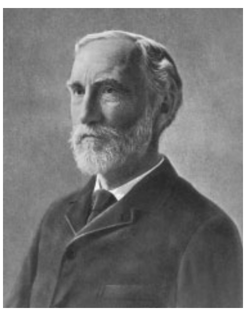
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \tag{15}$$

The companion formula for vectors in 2-space is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 \tag{16}$$

Motivated by the pattern in Formulas (15) and (16), we make the following definition.

> Although we derived Formula (15) and its 2-space companion under the assumption that $\mathbf{u}$ and $\mathbf{v}$ are nonzero, it turned out that these formulas are also applicable if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ (verify).



Josiah Willard Gibbs (1839–1903)

Historical Note The dot product notation was first introduced by the American physicist and mathematician J. Willard Gibbs in a pamphlet distributed to his students at Yale University in the 1880s. The product was originally written on the baseline, rather than centered as today, and was referred to as the *direct product*. Gibbs's pamphlet was eventually incorporated into a book entitled *Vector Analysis* that was published in 1901 and coauthored with one of his students. Gibbs made major contributions to the fields of thermodynamics and electromagnetic theory and is generally regarded as the greatest American physicist of the nineteenth century.

[*Image: SCIENCE SOURCE/Photo Researchers/ Getty Images*]

In words, to calculate the dot product (Euclidean inner product) multiply corresponding components and add the resulting products.

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{17}$$

EXAMPLE 6 Calculating Dot Products Using Components

(a) Use Formula (15) to compute the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ in Example 5.

(b) Calculate $\mathbf{u} \cdot \mathbf{v}$ for the following vectors in R^4:

$$\mathbf{u} = (-1, 3, 5, 7), \quad \mathbf{v} = (-3, -4, 1, 0)$$

Solution (a) The component forms of the vectors are $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = (0)(0) + (0)(2) + (1)(2) = 2$$

which agrees with the result obtained geometrically in Example 5.

Solution (b)

$$\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0) = -4$$

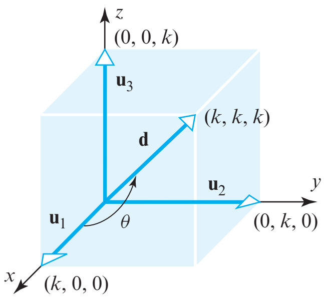


Figure 3.2.7

EXAMPLE 7 A Geometry Problem Solved Using Dot Product

Find the angle between a diagonal of a cube and one of its edges.

Solution Let k be the length of an edge and introduce a coordinate system as shown in Figure 3.2.7. If we let $\mathbf{u}_1 = (k, 0, 0)$, $\mathbf{u}_2 = (0, k, 0)$, and $\mathbf{u}_3 = (0, 0, k)$, then the vector

$$\mathbf{d} = (k, k, k) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. It follows from Formula (13) that the angle θ between $\mathbf{d}$ and the edge $\mathbf{u}_1$ satisfies

$$\cos\theta = \frac{\mathbf{u}_1 \cdot \mathbf{d}}{\|\mathbf{u}_1\|\|\mathbf{d}\|} = \frac{k^2}{(k)(\sqrt{3k^2})} = \frac{1}{\sqrt{3}}$$

With the help of a calculator we obtain

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ$$

Note that the angle θ obtained in Example 7 does not involve k. Why was this to be expected?

Algebraic Properties of the Dot Product

In the special case where $\mathbf{u} = \mathbf{v}$ in Definition 4, we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 \tag{18}$$

This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{19}$$

Dot products have many of the same algebraic properties as products of real numbers.

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

We will prove parts (*c*) and (*d*) and leave the other proofs as exercises.

Proof (c) Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Then

$$\begin{aligned} k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1v_1 + u_2v_2 + \cdots + u_nv_n) \\ &= (ku_1)v_1 + (ku_2)v_2 + \cdots + (ku_n)v_n = (k\mathbf{u}) \cdot \mathbf{v} \end{aligned}$$

Proof (d) The result follows from parts (*a*) and (*b*) of Theorem 3.2.1 and the fact that

$$\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + \cdots + v_nv_n = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 \quad ◀$$

The next theorem gives additional properties of dot products. The proofs can be obtained either by expressing the vectors in terms of components or by using the algebraic properties established in Theorem 3.2.2.

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then*:

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

We will show how Theorem 3.2.2 can be used to prove part (*b*) without breaking the vectors into components. The other proofs are left as exercises.

Proof (b)

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) && \text{[By symmetry]} \\ &= \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} && \text{[By distributivity]} \\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} && \text{[By symmetry]} \quad ◀ \end{aligned}$$

Formulas (18) and (19) together with Theorems 3.2.2 and 3.2.3 make it possible to manipulate expressions involving dot products using familiar algebraic techniques.

▶ EXAMPLE 8 Calculating with Dot Products

$$\begin{aligned} (\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v}) \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v}) \\ &= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2 \quad ◀ \end{aligned}$$

Cauchy–Schwarz Inequality and Angles in R^n

Our next objective is to extend to R^n the notion of "angle" between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. We will do this by starting with the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) \tag{20}$$

which we previously derived for nonzero vectors in R^2 and R^3. Since dot products and norms have been defined for vectors in R^n, it would seem that this formula has all the ingredients to serve as a *definition* of the angle θ between two vectors, $\mathbf{u}$ and $\mathbf{v}$, in R^n. However, there is a fly in the ointment, the problem being that the inverse cosine in Formula (20) is not defined unless its argument satisfies the inequalities

$$-1 \le \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \le 1 \tag{21}$$

Fortunately, these inequalities *do* hold for all nonzero vectors in R^n as a result of the following fundamental result known as the ***Cauchy–Schwarz inequality***.

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \le (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

We will omit the proof of this theorem because later in the text we will prove a more general version of which this will be a special case. Our goal for now will be to use this theorem to prove that the inequalities in (21) hold for all nonzero vectors in R^n. Once that is done we will have established all the results required to use Formula (20) as our *definition* of the angle between nonzero vectors **u** and **v** in R^n.

To prove that the inequalities in (21) hold for all nonzero vectors in R^n, divide both sides of Formula (22) by the product $\|\mathbf{u}\|\,\|\mathbf{v}\|$ to obtain

$$\frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \le 1 \quad \text{or equivalently} \quad \left|\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right| \le 1$$

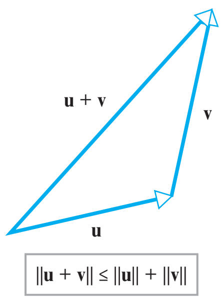
from which (21) follows.

Geometry in R^n

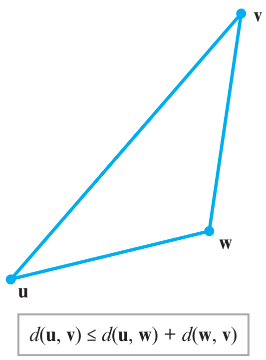
Earlier in this section we extended various concepts to R^n with the idea that familiar results that we can visualize in R^2 and R^3 might be valid in R^n as well. Here are two fundamental theorems from plane geometry whose validity extends to R^n:

- The sum of the lengths of two side of a triangle is at least as large as the third (Figure 3.2.8).
- The shortest distance between two points is a straight line (Figure 3.2.9).

The following theorem generalizes these theorems to R^n.



▲ Figure 3.2.8

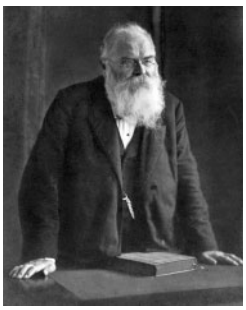


▲ Figure 3.2.9

THEOREM 3.2.5 *If* **u**, **v**, *and* **w** *are vectors in* R^n, *then*:

(*a*) $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(*b*) $d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ **[Triangle inequality for distances]**



Hermann Amandus Schwarz (1843–1921)

Viktor Yakovlevich Bunyakovsky (1804–1889)

Historical Note The Cauchy–Schwarz inequality is named in honor of the French mathematician Augustin Cauchy (see p. 121) and the German mathematician Hermann Schwarz. Variations of this inequality occur in many different settings and under various names. Depending on the context in which the inequality occurs, you may find it called Cauchy's inequality, the Schwarz inequality, or sometimes even the Bunyakovsky inequality, in recognition of the Russian mathematician who published his version of the inequality in 1859, about 25 years before Schwarz.

[*Images: © Rudolph Duehrkoop/ullstein bild/The Image Works* (*Schwarz*); *http://www-history.mcs.st-and.ac.uk/Biographies/Bunyakovsky.html* (*Bunyakovsky*)]

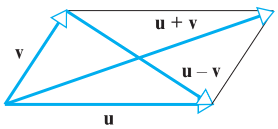
Proof (a)

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v}) \\
&= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 && \leftarrow \text{Property of absolute value} \\
&\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \leftarrow \text{Cauchy–Schwarz inequality} \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}
$$

This completes the proof since both sides of the inequality in part (*a*) are nonnegative.

Proof (b) It follows from part (*a*) and Formula (11) that

$$
\begin{aligned}
d(\mathbf{u},\mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| = \|(\mathbf{u}-\mathbf{w}) + (\mathbf{w}-\mathbf{v})\| \\
&\le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\| = d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}) \quad ◀
\end{aligned}
$$



▲ Figure 3.2.10

It is proved in plane geometry that for any parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of the four sides (Figure 3.2.10). The following theorem generalizes that result to R^n.

THEOREM 3.2.6 Parallelogram Equation for Vectors

If **u** *and* **v** *are vectors in* R^n, *then*

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \tag{24}$$

Proof

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
&= 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v}) \\
&= 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \quad ◀
\end{aligned}
$$

We could state and prove many more theorems from plane geometry that generalize to R^n, but the ones already given should suffice to convince you that R^n is not so different from R^2 and R^3 even though we cannot visualize it directly. The next theorem establishes a fundamental relationship between the dot product and norm in R^n.

THEOREM 3.2.7 *If* **u** *and* **v** *are vectors in* R^n *with the Euclidean inner product, then*

$$\mathbf{u}\cdot\mathbf{v} = \tfrac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \tfrac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 \tag{25}$$

Note that Formula (25) expresses the dot product in terms of norms.

Proof

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\
\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2
\end{aligned}
$$

from which (25) follows by simple algebra. ◀

Dot Products as Matrix Multiplication

There are various ways to express the dot product of vectors using matrix notation. The formulas depend on whether the vectors are expressed as row matrices or column matrices. Table 1 shows the possibilities.

Table 1

Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
u a row matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
u a column matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}$	$\mathbf{v}\mathbf{u}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$
u a row matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}$	$\mathbf{u}\mathbf{v}^T=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

Application of Dot Products to ISBN Numbers

Although the system has recently changed, most older books have been assigned a unique 10-digit number called an ***International Standard Book Number*** or ISBN. The first nine digits of this number are split into three groups—the first group representing the country or group of countries in which the book originates, the second identifying the publisher, and the third assigned to the book title itself. The tenth and final digit, called a ***check digit***, is computed from the first nine digits and is used to ensure that an electronic transmission of the ISBN, say over the Internet, occurs without error.

To explain how this is done, regard the first nine digits of the ISBN as a vector **b** in R^9, and let **a** be the vector

$$\mathbf{a}=(1,2,3,4,5,6,7,8,9)$$

Then the check digit c is computed using the following procedure:

1. Form the dot product $\mathbf{a}\cdot\mathbf{b}$.
2. Divide $\mathbf{a}\cdot\mathbf{b}$ by 11, thereby producing a remainder c that is an integer between 0 and 10, inclusive. The check digit is taken to be c, with the proviso that $c=10$ is written as X to avoid double digits.

For example, the ISBN of the brief edition of *Calculus*, sixth edition, by Howard Anton is

0-471-15307-9

which has a check digit of 9. This is consistent with the first nine digits of the ISBN, since

$$\mathbf{a}\cdot\mathbf{b}=(1,2,3,4,5,6,7,8,9)\cdot(0,4,7,1,1,5,3,0,7)=152$$

Dividing 152 by 11 produces a quotient of 13 and a remainder of 9, so the check digit is $c=9$. If an electronic order is placed for a book with a certain ISBN, then the warehouse can use the above procedure to verify that the check digit is consistent with the first nine digits, thereby reducing the possibility of a costly shipping error.

If A is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices, then it follows from the first row in Table 1 and properties of the transpose that

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^T A)\mathbf{u} = (A^T\mathbf{v})^T\mathbf{u} = \mathbf{u} \cdot A^T\mathbf{v}$$
$$\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T\mathbf{u} = (\mathbf{v}^T A^T)\mathbf{u} = \mathbf{v}^T(A^T\mathbf{u}) = A^T\mathbf{u} \cdot \mathbf{v}$$

The resulting formulas

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v} \tag{26}$$

$$\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v} \tag{27}$$

provide an important link between multiplication by an $n \times n$ matrix A and multiplication by A^T.

▶ EXAMPLE 9 Verifying that $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$

Suppose that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

from which we obtain

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10(0) + 5(5) = 11$$
$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

Thus, $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$ as guaranteed by Formula (26). We leave it for you to verify that Formula (27) also holds. ◀

A Dot Product View of Matrix Multiplication

Dot products provide another way of thinking about matrix multiplication. Recall that if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then the ijth entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

which is the dot product of the ith row vector of A

$$[a_{i1} \quad a_{i2} \quad \cdots \quad a_{ir}]$$

and the jth column vector of B

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}$$

Thus, if the row vectors of A are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column vectors of B are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix} \tag{28}$$

Exercise Set 3.2

▶ In Exercises 1–2, find the norm of $\mathbf{v}$, and a unit vector that is oppositely directed to $\mathbf{v}$. ◀

1. (a) $\mathbf{v} = (2, 2, 2)$ (b) $\mathbf{v} = (1, 0, 2, 1, 3)$

2. (a) $\mathbf{v} = (1, -1, 2)$ (b) $\mathbf{v} = (-2, 3, 3, -1)$

▶ In Exercises 3–4, evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$. ◀

3. (a) $\|\mathbf{u} + \mathbf{v}\|$ (b) $\|\mathbf{u}\| + \|\mathbf{v}\|$
(c) $\|-2\mathbf{u} + 2\mathbf{v}\|$ (d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$

4. (a) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$ (b) $\|\mathbf{u} - \mathbf{v}\|$
(c) $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$ (d) $\|\mathbf{u}\| - \|\mathbf{v}\|$

▶ In Exercises 5–6, evaluate the given expression with $\mathbf{u} = (-2, -1, 4, 5)$, $\mathbf{v} = (3, 1, -5, 7)$, and $\mathbf{w} = (-6, 2, 1, 1)$. ◀

5. (a) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$ (b) $\|3\mathbf{u}\| - 5\|\mathbf{v}\| + \|\mathbf{w}\|$
(c) $\|-\|\mathbf{u}\|\mathbf{v}\|$

6. (a) $\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$ (b) $\big\|\,\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\big\|$

7. Let $\mathbf{v} = (-2, 3, 0, 6)$. Find all scalars k such that $\|k\mathbf{v}\| = 5$.

8. Let $\mathbf{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|k\mathbf{v}\| = 4$.

▶ In Exercises 9–10, find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$. ◀

9. (a) $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (2, 2, -4)$
(b) $\mathbf{u} = (1, 1, 4, 6)$, $\mathbf{v} = (2, -2, 3, -2)$

10. (a) $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$
(b) $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

▶ In Exercises 11–12, find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°. ◀

11. (a) $\mathbf{u} = (3, 3, 3)$, $\mathbf{v} = (1, 0, 4)$
(b) $\mathbf{u} = (0, -2, -1, 1)$, $\mathbf{v} = (-3, 2, 4, 4)$

12. (a) $\mathbf{u} = (1, 2, -3, 0)$, $\mathbf{v} = (5, 1, 2, -2)$
(b) $\mathbf{u} = (0, 1, 1, 1, 2)$, $\mathbf{v} = (2, 1, 0, -1, 3)$

13. Suppose that a vector $\mathbf{a}$ in the xy-plane has a length of 9 units and points in a direction that is 120° counterclockwise from the positive x-axis, and a vector $\mathbf{b}$ in that plane has a length of 5 units and points in the positive y-direction. Find $\mathbf{a} \cdot \mathbf{b}$.

14. Suppose that a vector $\mathbf{a}$ in the xy-plane points in a direction that is 47° counterclockwise from the positive x-axis, and a vector $\mathbf{b}$ in that plane points in a direction that is 43° clockwise from the positive x-axis. What can you say about the value of $\mathbf{a} \cdot \mathbf{b}$?

▶ In Exercises 15–16, determine whether the expression makes sense mathematically. If not, explain why. ◀

15. (a) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
(c) $\|\mathbf{u} \cdot \mathbf{v}\|$ (d) $(\mathbf{u} \cdot \mathbf{v}) - \|\mathbf{u}\|$

16. (a) $\|\mathbf{u}\| \cdot \|\mathbf{v}\|$ (b) $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{w}$
(c) $(\mathbf{u} \cdot \mathbf{v}) - k$ (d) $k \cdot \mathbf{u}$

▶ In Exercises 17–18, verify that the Cauchy–Schwarz inequality holds. ◀

17. (a) $\mathbf{u} = (-3, 1, 0)$, $\mathbf{v} = (2, -1, 3)$
(b) $\mathbf{u} = (0, 2, 2, 1)$, $\mathbf{v} = (1, 1, 1, 1)$

18. (a) $\mathbf{u} = (4, 1, 1)$, $\mathbf{v} = (1, 2, 3)$
(b) $\mathbf{u} = (1, 2, 1, 2, 3)$, $\mathbf{v} = (0, 1, 1, 5, -2)$

19. Let $\mathbf{r}_0 = (x_0, y_0)$ be a fixed vector in R^2. In each part, describe in words the set of all vectors $\mathbf{r} = (x, y)$ that satisfy the stated condition.
(a) $\|\mathbf{r} - \mathbf{r}_0\| = 1$ (b) $\|\mathbf{r} - \mathbf{r}_0\| \le 1$ (c) $\|\mathbf{r} - \mathbf{r}_0\| > 1$

20. Repeat the directions of Exercise 19 for vectors $\mathbf{r} = (x, y, z)$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$ in R^3.

▶ **Exercises 21–25** The direction of a nonzero vector $\mathbf{v}$ in an xyz-coordinate system is completely determined by the angles α, β, and γ between $\mathbf{v}$ and the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Figure Ex-21). These are called the ***direction angles*** of $\mathbf{v}$, and their cosines are called the ***direction cosines*** of $\mathbf{v}$. ◀

21. Use Formula (13) to show that the direction cosines of a vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 are

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

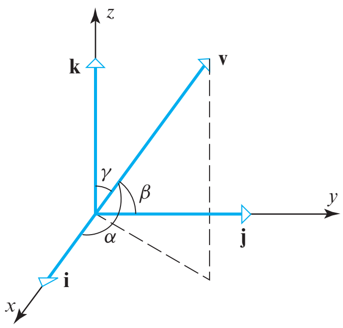


Figure Ex-21

22. Use the result in Exercise 21 to show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

23. Show that two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in R^3 are orthogonal if and only if their direction cosines satisfy

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$$

24. The accompanying figure shows a cube.

(a) Find the angle between the vectors **d** and **u** to the nearest degree.

(b) Make a conjecture about the angle between the vectors **d** and **v**, and confirm your conjecture by computing the angle.

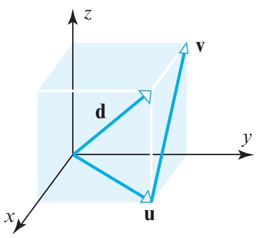


Figure Ex-24

25. Estimate, to the nearest degree, the angles that a diagonal of a box with dimensions 10 cm × 15 cm × 25 cm makes with the edges of the box.

26. If $\|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$, what are the largest and smallest values possible for $\|\mathbf{v} - \mathbf{w}\|$? Give a geometric explanation of your results.

27. What can you say about two nonzero vectors, **u** and **v**, that satisfy the equation $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$?

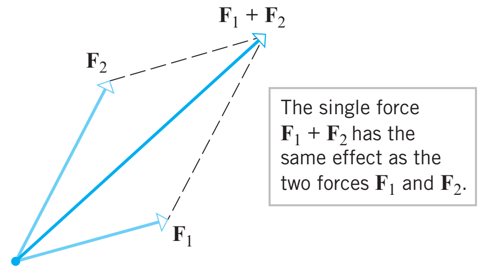
28. (a) What relationship must hold for the point $\mathbf{p} = (a, b, c)$ to be equidistant from the origin and the xz-plane? Make sure that the relationship you state is valid for positive and negative values of a, b, and c.

(b) What relationship must hold for the point $\mathbf{p} = (a, b, c)$ to be farther from the origin than from the xz-plane? Make sure that the relationship you state is valid for positive and negative values of a, b, and c.

29. State a procedure for finding a vector of a specified length m that points in the same direction as a given vector **v**.

30. Under what conditions will the triangle inequality (Theorem 3.2.5a) be an equality? Explain your answer geometrically.

▶ **Exercises 31–32** The effect that a force has on an object depends on the magnitude of the force and the direction in which it is applied. Thus, forces can be regarded as vectors and represented as arrows in which the length of the arrow specifies the magnitude of the force, and the direction of the arrow specifies the direction in which the force is applied. It is a fact of physics that force vectors obey the parallelogram law in the sense that if two force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied at a point on an object, then the effect is the same as if the single force $\mathbf{F}_1 + \mathbf{F}_2$ (called the ***resultant***) were applied at that point (see accompanying figure). Forces are commonly measured in units called pounds-force (abbreviated lbf) or Newtons (abbreviated N). ◀



31. A particle is said to be in ***static equilibrium*** if the resultant of all forces applied to it is zero. For the forces in the accompanying figure, find the resultant **F** that must be applied to the indicated point to produce static equilibrium. Describe **F** by giving its magnitude and the angle in degrees that it makes with the positive x-axis.

32. Follow the directions of Exercise 31.

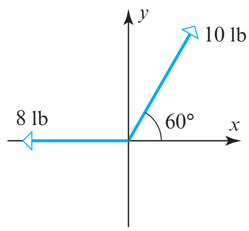


Figure Ex-31

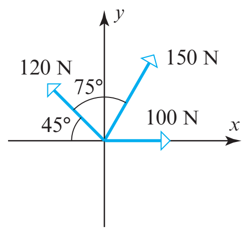


Figure Ex-32

Working with Proofs

33. Prove parts (a) and (b) of Theorem 3.2.1.

34. Prove parts (a) and (c) of Theorem 3.2.3.

35. Prove parts (d) and (e) of Theorem 3.2.3.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) If each component of a vector in R^3 is doubled, the norm of that vector is doubled.

(b) In R^2, the vectors of norm 5 whose initial points are at the origin have terminal points lying on a circle of radius 5 centered at the origin.

(c) Every vector in R^n has a positive norm.

(d) If $\mathbf{v}$ is a nonzero vector in R^n, there are exactly two unit vectors that are parallel to $\mathbf{v}$.

(e) If $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 1$, and $\mathbf{u} \cdot \mathbf{v} = 1$, then the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$ radians.

(f) The expressions $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$ and $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ are both meaningful and equal to each other.

(g) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(h) If $\mathbf{u} \cdot \mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

(i) In R^2, if $\mathbf{u}$ lies in the first quadrant and $\mathbf{v}$ lies in the third quadrant, then $\mathbf{u} \cdot \mathbf{v}$ cannot be positive.

(j) For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in R^n, we have

$$\|\mathbf{u} + \mathbf{v} + \mathbf{w}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\|$$

Working with Technology

T1. Let $\mathbf{u}$ be a vector in R^{100} whose ith component is i, and let $\mathbf{v}$ be the vector in R^{100} whose ith component is $1/(i+1)$. Find the dot product of $\mathbf{u}$ and $\mathbf{v}$.

T2. Find, to the nearest degree, the angles that a diagonal of a box with dimensions 10 cm × 11 cm × 25 cm makes with the edges of the box.

3.3 Orthogonality

In the last section we defined the notion of "angle" between vectors in R^n. In this section we will focus on the notion of "perpendicularity." Perpendicular vectors in R^n play an important role in a wide variety of applications.

Orthogonal Vectors

Recall from Formula (20) in the previous section that the angle θ between two *nonzero* vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is defined by the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

It follows from this that $\theta = \pi/2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. Thus, we make the following definition.

DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

EXAMPLE 1 Orthogonal Vectors

(a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4.

(b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3. Show that each ordered pair of vectors in S is orthogonal.

Solution (a) The vectors are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Solution (b) It suffices to show that

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$$

> Using the computations in R^3 as a model, you should be able to see that each ordered pair of standard unit vectors in R^n is orthogonal.

because it will follow automatically from the symmetry property of the dot product that

$$\mathbf{j}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{j} = \mathbf{0}$$

Although the orthogonality of the vectors in S is evident geometrically from Figure 3.2.2, it is confirmed algebraically by the computations

$$\mathbf{i}\cdot\mathbf{j} = (1, 0, 0)\cdot(0, 1, 0) = 0$$
$$\mathbf{i}\cdot\mathbf{k} = (1, 0, 0)\cdot(0, 0, 1) = 0$$
$$\mathbf{j}\cdot\mathbf{k} = (0, 1, 0)\cdot(0, 0, 1) = 0 \quad ◀$$

Lines and Planes Determined by Points and Normals

One learns in analytic geometry that a line in R^2 is determined uniquely by its slope and one of its points, and that a plane in R^3 is determined uniquely by its "inclination" and one of its points. One way of specifying slope and inclination is to use a *nonzero* vector $\mathbf{n}$, called a ***normal***, that is orthogonal to the line or plane in question. For example, Figure 3.3.1 shows the line through the point $P_0(x_0, y_0)$ that has normal $\mathbf{n} = (a, b)$ and the plane through the point $P_0(x_0, y_0, z_0)$ that has normal $\mathbf{n} = (a, b, c)$. Both the line and the plane are represented by the vector equation

$$\mathbf{n}\cdot\overrightarrow{P_0P} = 0 \tag{1}$$

where P is either an arbitrary point (x, y) on the line or an arbitrary point (x, y, z) in the plane. The vector $\overrightarrow{P_0P}$ can be expressed in terms of components as

> Formula (1) is called the ***point-normal*** form of a line or plane and Formulas (2) and (3) the ***component*** forms.

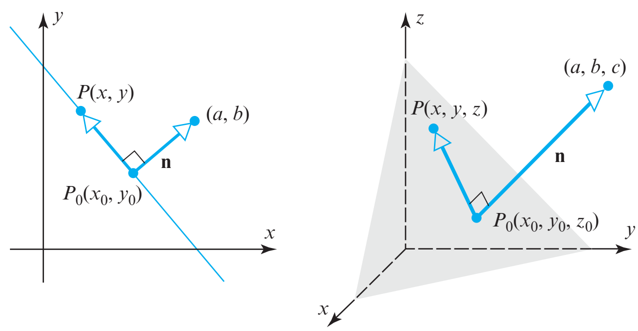
$$\overrightarrow{P_0P} = (x - x_0, y - y_0) \quad \textbf{[line]}$$
$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0) \quad \textbf{[plane]}$$

Thus, Equation (1) can be written as

$$a(x - x_0) + b(y - y_0) = 0 \quad \textbf{[line]} \tag{2}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \textbf{[plane]} \tag{3}$$

These are called the ***point-normal*** equations of the line and plane.



▶ Figure 3.3.1

▶ EXAMPLE 2 Point-Normal Equations

It follows from (2) that in R^2 the equation

$$6(x - 3) + (y + 7) = 0$$

represents the line through the point $(3, -7)$ with normal $\mathbf{n} = (6, 1)$; and it follows from (3) that in R^3 the equation

$$4(x - 3) + 2y - 5(z - 7) = 0$$

represents the plane through the point $(3, 0, 7)$ with normal $\mathbf{n} = (4, 2, -5)$. ◀

When convenient, the terms in Equations (2) and (3) can be multiplied out and the constants combined. This leads to the following theorem.

THEOREM 3.3.1

(*a*) *If a and b are constants that are not both zero, then an equation of the form*

$$ax + by + c = 0 \tag{4}$$

represents a line in R^2 *with normal* $\mathbf{n} = (a, b)$.

(*b*) *If* a, b, *and* c *are constants that are not all zero, then an equation of the form*

$$ax + by + cz + d = 0 \tag{5}$$

represents a plane in R^3 *with normal* $\mathbf{n} = (a, b, c)$.

EXAMPLE 3 Vectors Orthogonal to Lines and Planes Through the Origin

(a) The equation $ax + by = 0$ represents a line through the origin in R^2. Show that the vector $\mathbf{n}_1 = (a, b)$ formed from the coefficients of the equation is orthogonal to the line, that is, orthogonal to every vector along the line.

(b) The equation $ax + by + cz = 0$ represents a plane through the origin in R^3. Show that the vector $\mathbf{n}_2 = (a, b, c)$ formed from the coefficients of the equation is orthogonal to the plane, that is, orthogonal to every vector that lies in the plane.

Solution We will solve both problems together. The two equations can be written as

$$(a, b) \cdot (x, y) = 0 \quad \text{and} \quad (a, b, c) \cdot (x, y, z) = 0$$

or, alternatively, as

$$\mathbf{n}_1 \cdot (x, y) = 0 \quad \text{and} \quad \mathbf{n}_2 \cdot (x, y, z) = 0$$

These equations show that $\mathbf{n}_1$ is orthogonal to every vector (x, y) on the line and that $\mathbf{n}_2$ is orthogonal to every vector (x, y, z) in the plane (Figure 3.3.1). ◀

Recall that

$$ax + by = 0 \quad \text{and} \quad ax + by + cz = 0$$

are called *homogeneous equations*. Example 3 illustrates that homogeneous equations in two or three unknowns can be written in the vector form

$$\mathbf{n} \cdot \mathbf{x} = 0 \tag{6}$$

where $\mathbf{n}$ is the vector of coefficients and $\mathbf{x}$ is the vector of unknowns. In R^2 this is called the ***vector form of a line*** through the origin, and in R^3 it is called the ***vector form of a plane*** through the origin.

Referring to Table 1 of Section 3.2, in what other ways can you write (6) if $\mathbf{n}$ and $\mathbf{x}$ are expressed in matrix form?

Orthogonal Projections

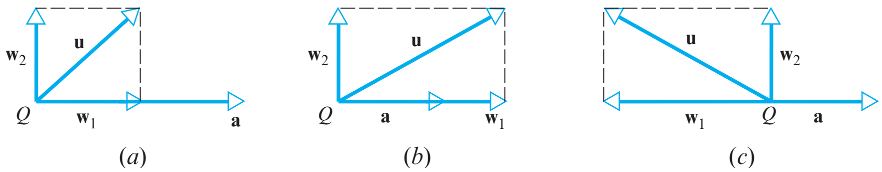
In many applications it is necessary to "decompose" a vector $\mathbf{u}$ into a sum of two terms, one term being a scalar multiple of a specified nonzero vector $\mathbf{a}$ and the other term being orthogonal to $\mathbf{a}$. For example, if $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^2 that are positioned so their initial points coincide at a point Q, then we can create such a decomposition as follows (Figure 3.3.2):

- Drop a perpendicular from the tip of **u** to the line through **a**.
- Construct the vector $\mathbf{w}_1$ from Q to the foot of the perpendicular.
- Construct the vector $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

Since

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$$

we have decomposed **u** into a sum of two orthogonal vectors, the first term being a scalar multiple of **a** and the second being orthogonal to **a**.



▲ Figure 3.3.2 Three possible cases.

The following theorem shows that the foregoing results, which we illustrated using vectors in R^2, apply as well in R^n.

THEOREM 3.3.2 Projection Theorem

If **u** *and* **a** *are vectors in* R^n, *and if* $\mathbf{a} \neq 0$, *then* **u** *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* **a** *and* $\mathbf{w}_2$ *is orthogonal to* **a**.

Proof Since the vector $\mathbf{w}_1$ is to be a scalar multiple of **a**, it must have the form

$$\mathbf{w}_1 = k\mathbf{a} \tag{7}$$

Our goal is to find a value of the scalar k and a vector $\mathbf{w}_2$ that is orthogonal to **a** such that

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 \tag{8}$$

We can determine k by using (7) to rewrite (8) as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = k\mathbf{a} + \mathbf{w}_2$$

and then applying Theorems 3.2.2 and 3.2.3 to obtain

$$\mathbf{u} \cdot \mathbf{a} = (k\mathbf{a} + \mathbf{w}_2) \cdot \mathbf{a} = k\|\mathbf{a}\|^2 + (\mathbf{w}_2 \cdot \mathbf{a}) \tag{9}$$

Since $\mathbf{w}_2$ is to be orthogonal to **a**, the last term in (9) must be 0, and hence k must satisfy the equation

$$\mathbf{u} \cdot \mathbf{a} = k\|\mathbf{a}\|^2$$

from which we obtain

$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}$$

as the only possible value for k. The proof can be completed by rewriting (8) as

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \mathbf{u} - k\mathbf{a} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

and then confirming that $\mathbf{w}_2$ is orthogonal to **a** by showing that $\mathbf{w}_2 \cdot \mathbf{a} = 0$ (we leave the details for you). ◀

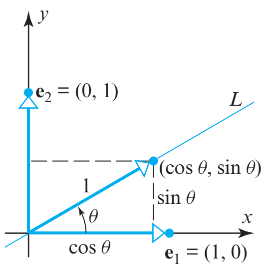
The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the Projection Theorem have associated names—the vector $\mathbf{w}_1$ is called the ***orthogonal projection of* u *on* a** or sometimes ***the vector component of* u *along* a**, and the vector $\mathbf{w}_2$ is called the ***vector component of* u *orthogonal to* a**. The vector $\mathbf{w}_1$ is commonly denoted by the symbol $\text{proj}_{\mathbf{a}}\mathbf{u}$, in which case it follows from (8) that $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$. In summary,

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad (\textit{vector component of } \mathbf{u} \textit{ along } \mathbf{a}) \tag{10}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad (\textit{vector component of } \mathbf{u} \textit{ orthogonal to } \mathbf{a}) \tag{11}$$

▶ EXAMPLE 4 Orthogonal Projection on a Line

Find the orthogonal projections of the vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ on the line L that makes an angle θ with the positive x-axis in R^2.



▲ Figure 3.3.3

Solution As illustrated in Figure 3.3.3, $\mathbf{a} = (\cos\theta, \sin\theta)$ is a unit vector along the line L, so our first problem is to find the orthogonal projection of $\mathbf{e}_1$ along $\mathbf{a}$. Since

$$\|\mathbf{a}\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1 \quad \text{and} \quad \mathbf{e}_1 \cdot \mathbf{a} = (1, 0) \cdot (\cos\theta, \sin\theta) = \cos\theta$$

it follows from Formula (10) that this projection is

$$\text{proj}_{\mathbf{a}}\mathbf{e}_1 = \frac{\mathbf{e}_1 \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = (\cos\theta)(\cos\theta, \sin\theta) = (\cos^2\theta, \sin\theta\cos\theta)$$

Similarly, since $\mathbf{e}_2 \cdot \mathbf{a} = (0, 1) \cdot (\cos\theta, \sin\theta) = \sin\theta$, it follows from Formula (10) that

$$\text{proj}_{\mathbf{a}}\mathbf{e}_2 = \frac{\mathbf{e}_2 \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = (\sin\theta)(\cos\theta, \sin\theta) = (\sin\theta\cos\theta, \sin^2\theta)$$

▶ EXAMPLE 5 Vector Component of u Along a

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

Solution

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$
$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

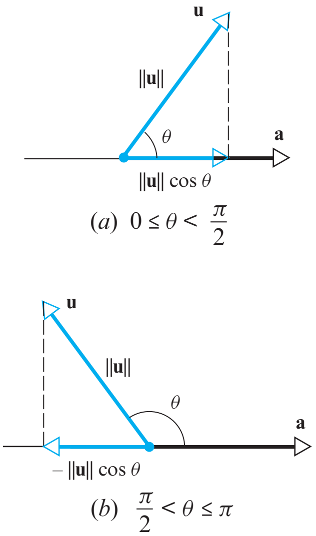
Thus the vector component of $\mathbf{u}$ along $\mathbf{a}$ is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \tfrac{15}{21}(4, -1, 2) = \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right)$$

and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right) = \left(-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{11}{7}\right)$$

As a check, you may wish to verify that the vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{a}$ are perpendicular by showing that their dot product is zero. ◀



▲ Figure 3.3.4

Sometimes we will be more interested in the *norm* of the vector component of **u** along **a** than in the vector component itself. A formula for this norm can be derived as follows:

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \left\| \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \right\| = \left| \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2} \right| \|\mathbf{a}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\|$$

where the second equality follows from part (*c*) of Theorem 3.2.1 and the third from the fact that $\|\mathbf{a}\|^2 > 0$. Thus,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} \tag{12}$$

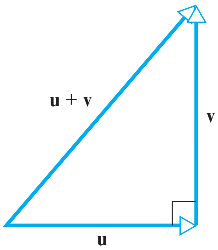
If θ denotes the angle between **u** and **a**, then $\mathbf{u}\cdot\mathbf{a} = \|\mathbf{u}\|\|\mathbf{a}\|\cos\theta$, so (12) can also be written as

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \|\mathbf{u}\||\cos\theta| \tag{13}$$

(Verify.) A geometric interpretation of this result is given in Figure 3.3.4.

The Theorem of Pythagoras

In Section 3.2 we found that many theorems about vectors in R^2 and R^3 also hold in R^n. Another example of this is the following generalization of the Theorem of Pythagoras (Figure 3.3.5).



▲ Figure 3.3.5

THEOREM 3.3.3 Theorem of Pythagoras in R^n

If **u** *and* **v** *are orthogonal vectors in* R^n *with the Euclidean inner product, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \tag{14}$$

Proof Since **u** and **v** are orthogonal, we have $\mathbf{u}\cdot\mathbf{v} = 0$, from which it follows that

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad ◀$$

▶ EXAMPLE 6 Theorem of Pythagoras in R^4

We showed in Example 1 that the vectors

$$\mathbf{u} = (-2, 3, 1, 4) \quad \text{and} \quad \mathbf{v} = (1, 2, 0, -1)$$

are orthogonal. Verify the Theorem of Pythagoras for these vectors.

Solution We leave it for you to confirm that

$$\mathbf{u}+\mathbf{v} = (-1, 5, 1, 3)$$
$$\|\mathbf{u}+\mathbf{v}\|^2 = 36$$
$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 30 + 6$$

Thus, $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ ◀

OPTIONAL

Distance Problems

We will now show how orthogonal projections can be used to solve the following three distance problems:

Problem 1. Find the distance between a point and a line in R^2.

Problem 2. Find the distance between a point and a plane in R^3.

Problem 3. Find the distance between two parallel planes in R^3.

A method for solving the first two problems is provided by the next theorem. Since the proofs of the two parts are similar, we will prove part (b) and leave part (a) as an exercise.

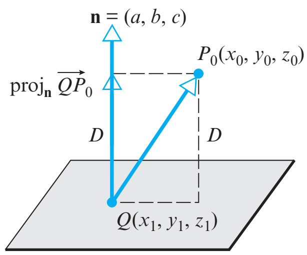
THEOREM 3.3.4

(a) *In R^2 the distance D between the point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$ is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(b) *In R^3 the distance D between the point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$



Distance from P_0 to plane.

▲ Figure 3.3.6

Proof (b) The underlying idea of the proof is illustrated in Figure 3.3.6. As shown in that figure, let $Q(x_1, y_1, z_1)$ be any point in the plane, and let $\mathbf{n} = (a, b, c)$ be a normal vector to the plane that is positioned with its initial point at Q. It is now evident that the distance D between P_0 and the plane is simply the length (or norm) of the orthogonal projection of the vector $\overrightarrow{QP_0}$ on $\mathbf{n}$, which by Formula (12) is

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{QP_0}\| = \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

But

$$\overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$$

$$\overrightarrow{QP_0} \cdot \mathbf{n} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$$

$$\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}$$

Thus

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \tag{17}$$

Since the point $Q(x_1, y_1, z_1)$ lies in the given plane, its coordinates satisfy the equation of that plane; thus

$$ax_1 + by_1 + cz_1 + d = 0$$

or

$$d = -ax_1 - by_1 - cz_1$$

Substituting this expression in (17) yields (16). ◀

▶ EXAMPLE 7 Distance Between a Point and a Plane

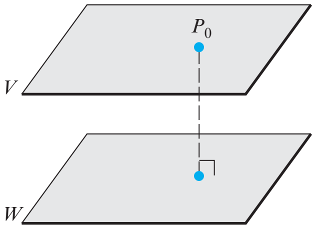
Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

Solution Since the distance formulas in Theorem 3.3.4 require that the equations of the line and plane be written with zero on the right side, we first need to rewrite the equation of the plane as

$$2x - 3y + 6z + 1 = 0$$

from which we obtain

$$D = \frac{|2(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7} \quad ◀$$



▲ Figure 3.3.7 The distance between the parallel planes V and W is equal to the distance between P_0 and W.

The third distance problem posed above is to find the distance between two parallel planes in R^3. As suggested in Figure 3.3.7, the distance between a plane V and a plane W can be obtained by finding any point P_0 in one of the planes, and computing the distance between that point and the other plane. Here is an example.

▶ EXAMPLE 8 Distance Between Parallel Planes

The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $(1, 2, -2)$ and $(2, 4, -4)$, are parallel vectors. Find the distance between these planes.

Solution To find the distance D between the planes, we can select an arbitrary point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (16), the distance between P_0 and the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|2(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$ ◀

Exercise Set 3.3

▶ In Exercises 1–2, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors. ◀

1. (a) $\mathbf{u} = (6, 1, 4)$, $\mathbf{v} = (2, 0, -3)$

(b) $\mathbf{u} = (0, 0, -1)$, $\mathbf{v} = (1, 1, 1)$

(c) $\mathbf{u} = (3, -2, 1, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$

(d) $\mathbf{u} = (5, -4, 0, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$

2. (a) $\mathbf{u} = (2, 3)$, $\mathbf{v} = (5, -7)$

(b) $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (0, 0, 0)$

(c) $\mathbf{u} = (1, -5, 4)$, $\mathbf{v} = (3, 3, 3)$

(d) $\mathbf{u} = (4, 1, -2, 5)$, $\mathbf{v} = (-1, 5, 3, 1)$

▶ In Exercises 3–6, find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal. ◀

3. $P(-1, 3, -2)$; $\mathbf{n} = (-2, 1, -1)$

4. $P(1, 1, 4)$; $\mathbf{n} = (1, 9, 8)$ **5.** $P(2, 0, 0)$; $\mathbf{n} = (0, 0, 2)$

6. $P(0, 0, 0)$; $\mathbf{n} = (1, 2, 3)$

▶ In Exercises 7–10, determine whether the given planes are parallel. ◀

7. $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$

8. $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$

9. $2y = 8x - 4z + 5$ and $x = \frac{1}{2}z + \frac{1}{4}y$

10. $(-4, 1, 2) \cdot (x, y, z) = 0$ and $(8, -2, -4) \cdot (x, y, z) = 0$

▶ In Exercises 11–12, determine whether the given planes are perpendicular. ◀

11. $3x - y + z - 4 = 0$, $x + 2z = -1$

12. $x - 2y + 3z = 4$, $-2x + 5y + 4z = -1$

▶ In Exercises 13–14, find $\|\text{proj}_{\mathbf{a}}\mathbf{u}\|$. ◀

13. (a) $\mathbf{u} = (1, -2)$, $\mathbf{a} = (-4, -3)$

(b) $\mathbf{u} = (3, 0, 4)$, $\mathbf{a} = (2, 3, 3)$

14. (a) $\mathbf{u} = (5, 6)$, $\mathbf{a} = (2, -1)$

(b) $\mathbf{u} = (3, -2, 6)$, $\mathbf{a} = (1, 2, -7)$

▶ In Exercises 15–20, find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$. ◀

15. $\mathbf{u} = (6, 2)$, $\mathbf{a} = (3, -9)$ **16.** $\mathbf{u} = (-1, -2)$, $\mathbf{a} = (-2, 3)$

17. $\mathbf{u} = (3, 1, -7)$, $\mathbf{a} = (1, 0, 5)$

18. $\mathbf{u} = (2, 0, 1)$, $\mathbf{a} = (1, 2, 3)$

19. $\mathbf{u} = (2, 1, 1, 2)$, $\mathbf{a} = (4, -4, 2, -2)$

20. $\mathbf{u} = (5, 0, -3, 7)$, $\mathbf{a} = (2, 1, -1, -1)$

▶ In Exercises 21–24, find the distance between the point and the line. ◀

21. $(-3, 1)$; $4x + 3y + 4 = 0$

22. $(-1, 4)$; $x - 3y + 2 = 0$

23. $(2, -5)$; $y = -4x + 2$

24. $(1, 8)$; $3x + y = 5$

▶ In Exercises 25–26, find the distance between the point and the plane. ◀

25. $(3, 1, -2)$; $x + 2y - 2z = 4$

26. $(-1, -1, 2)$; $2x + 5y - 6z = 4$

▶ In Exercises 27–28, find the distance between the given parallel planes. ◀

27. $2x - y - z = 5$ and $-4x + 2y + 2z = 12$

28. $2x - y + z = 1$ and $2x - y + z = -1$

29. Find a unit vector that is orthogonal to both $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (0, 1, 1)$.

30. (a) Show that $\mathbf{v} = (a, b)$ and $\mathbf{w} = (-b, a)$ are orthogonal vectors.

(b) Use the result in part (a) to find two vectors that are orthogonal to $\mathbf{v} = (2, -3)$.

(c) Find two unit vectors that are orthogonal to $\mathbf{v} = (-3, 4)$.

31. Do the points $A(1, 1, 1)$, $B(-2, 0, 3)$, and $C(-3, -1, 1)$ form the vertices of a right triangle? Explain.

32. Repeat Exercise 31 for the points $A(3, 0, 2)$, $B(4, 3, 0)$, and $C(8, 1, -1)$.

33. Show that if $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is orthogonal to $k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for all scalars k_1 and k_2.

34. Is it possible to have $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a}$? Explain.

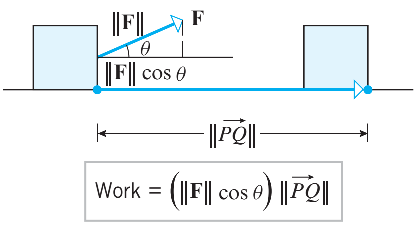
▶ **Exercises 35–37** In physics and engineering the ***work*** W performed by a *constant force* $\mathbf{F}$ applied in the *direction of motion* to an object moving a distance d on a straight line is defined to be

$$W = \|\mathbf{F}\|d \quad \text{(force magnitude times distance)}$$

In the case where the applied force is constant but makes an angle θ with the direction of motion, and where the object moves along a line from a point P to a point Q, we call $\overrightarrow{PQ}$ the ***displacement*** and define the work performed by the force to be

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \cos\theta$$

(see accompanying figure). Common units of work are ft-lb (foot pounds) or Nm (Newton meters). ◀

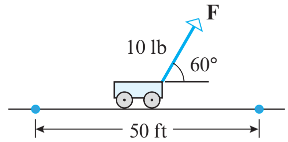


35. Show that the work performed by a constant force (not necessarily in the direction of motion) can be expressed as

$$W = \pm\|\overrightarrow{PQ}\|\,\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|$$

and explain when the $+$ sign should be used and when the $-$ sign should be used.

36. As illustrated in the accompanying figure, a wagon is pulled horizontally by exerting a force of 10 lb on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon 50 ft?



37. A sailboat travels 100 m due north while the wind exerts a force of 500 N toward the northeast. How much work does the wind do?

Working with Proofs

38. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in 2- or 3-space, and let $k = \|\mathbf{u}\|$ and $l = \|\mathbf{v}\|$. Prove that the vector $\mathbf{w} = l\mathbf{u} + k\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

39. Prove part (a) of Theorem 3.3.4.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) The vectors $(3, -1, 2)$ and $(0, 0, 0)$ are orthogonal.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then for all nonzero scalars k and m, $k\mathbf{u}$ and $m\mathbf{v}$ are orthogonal vectors.

(c) The orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$ is perpendicular to the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

(d) If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors, then for every nonzero vector $\mathbf{u}$, we have

$$\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{b}}(\mathbf{u})) = \mathbf{0}$$

(e) If $\mathbf{a}$ and $\mathbf{u}$ are nonzero vectors, then

$$\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{a}}(\mathbf{u})) = \text{proj}_{\mathbf{a}}(\mathbf{u})$$

(f) If the relationship

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{a}}\mathbf{v}$$

holds for some nonzero vector $\mathbf{a}$, then $\mathbf{u} = \mathbf{v}$.

(g) For all vectors $\mathbf{u}$ and $\mathbf{v}$, it is true that

$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$$

Working with Technology

T1. Find the lengths of the sides and the interior angles of the triangle in R^4 whose vertices are

$$P(2, 4, 2, 4, 2), \quad Q(6, 4, 4, 4, 6), \quad R(5, 7, 5, 7, 2)$$

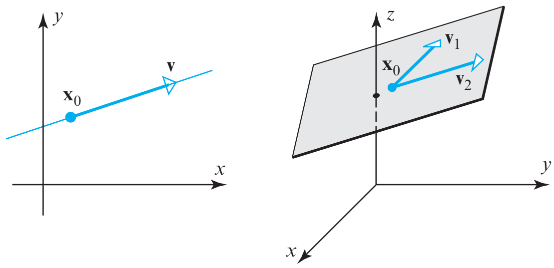
T2. Express the vector $\mathbf{u} = (2, 3, 1, 2)$ in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a} = (-1, 0, 2, 1)$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

3.4 The Geometry of Linear Systems

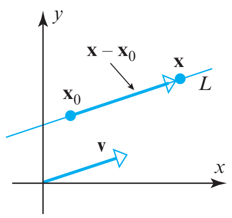
In this section we will use parametric and vector methods to study general systems of linear equations. This work will enable us to interpret solution sets of linear systems with n unknowns as geometric objects in R^n just as we interpreted solution sets of linear systems with two and three unknowns as points, lines, and planes in R^2 and R^3.

Vector and Parametric Equations of Lines in R^2 and R^3

In the last section we derived equations of lines and planes that are determined by a point and a normal vector. However, there are other useful ways of specifying lines and planes. For example, a unique line in R^2 or R^3 is determined by a point $\mathbf{x}_0$ on the line and a nonzero vector $\mathbf{v}$ parallel to the line, and a unique plane in R^3 is determined by a point $\mathbf{x}_0$ in the plane and two noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ parallel to the plane. The best way to visualize this is to translate the vectors so their initial points are at $\mathbf{x}_0$ (Figure 3.4.1).



▶ Figure 3.4.1



▲ Figure 3.4.2

Let us begin by deriving an equation for the line L that contains the point $\mathbf{x}_0$ and is parallel to $\mathbf{v}$. If $\mathbf{x}$ is a general point on such a line, then, as illustrated in Figure 3.4.2, the vector $\mathbf{x} - \mathbf{x}_0$ will be some scalar multiple of $\mathbf{v}$, say

$$\mathbf{x} - \mathbf{x}_0 = t\mathbf{v} \quad \text{or equivalently} \quad \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

As the variable t (called a ***parameter***) varies from $-\infty$ to ∞, the point $\mathbf{x}$ traces out the line L. Accordingly, we have the following result.

> Although it is not stated explicitly, it is understood in Formulas (1) and (2) that the parameter t varies from $-\infty$ to ∞. This applies to all vector and parametric equations in this text except where stated otherwise.

THEOREM 3.4.1 *Let L be the line in R^2 or R^3 that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{1}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v} \tag{2}$$

Vector and Parametric Equations of Planes in R^3

Next we will derive an equation for the plane W that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. As shown in Figure 3.4.3, if $\mathbf{x}$ is any point in the plane, then by forming suitable scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$, say $t_1\mathbf{v}_1$ and $t_2\mathbf{v}_2$, we can create a parallelogram with diagonal $\mathbf{x} - \mathbf{x}_0$ and adjacent sides $t_1\mathbf{v}_1$ and $t_2\mathbf{v}_2$. Thus, we have

$$\mathbf{x} - \mathbf{x}_0 = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad \text{or equivalently} \quad \mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

As the variables t_1 and t_2 (called ***parameters***) vary independently from $-\infty$ to ∞, the point $\mathbf{x}$ varies over the entire plane W. In summary, we have the following result.

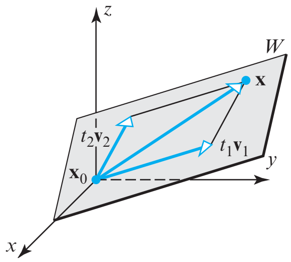


Figure 3.4.3

THEOREM 3.4.2 *Let W be the plane in R^3 that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by*

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{3}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{4}$$

Remark Observe that the line through $\mathbf{x}_0$ represented by Equation (1) is the translation by $\mathbf{x}_0$ of the line through the origin represented by Equation (2) and that the plane through $\mathbf{x}_0$ represented by Equation (3) is the translation by $\mathbf{x}_0$ of the plane through the origin represented by Equation (4) (Figure 3.4.4).

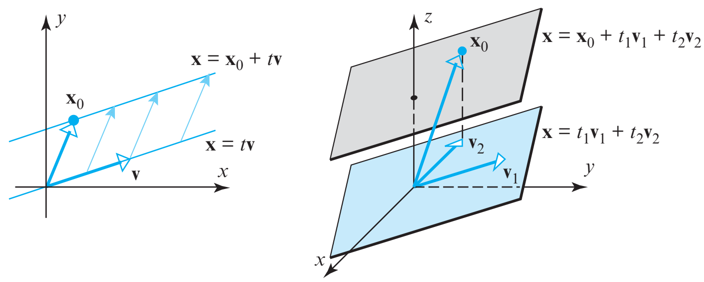


Figure 3.4.4

Motivated by the forms of Formulas (1) to (4), we can extend the notions of line and plane to R^n by making the following definitions.

DEFINITION 1 If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{5}$$

defines the ***line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$***. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

DEFINITION 2 If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{6}$$

defines the ***plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$***. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

Equations (5) and (6) are called ***vector forms*** of a line and plane in R^n. If the vectors in these equations are expressed in terms of their components and the corresponding components on each side are equated, then the resulting equations are called ***parametric equations*** of the line and plane. Here are some examples.

EXAMPLE 1 Vector and Parametric Equations of Lines in R^2 and R^3

(a) Find a vector equation and parametric equations of the line in R^2 that passes through the origin and is parallel to the vector $\mathbf{v} = (-2, 3)$.

(b) Find a vector equation and parametric equations of the line in R^3 that passes through the point $P_0(1, 2, -3)$ and is parallel to the vector $\mathbf{v} = (4, -5, 1)$.

(c) Use the vector equation obtained in part (b) to find two points on the line that are different from P_0.

Solution (a) It follows from (5) with $\mathbf{x}_0 = \mathbf{0}$ that a vector equation of the line is $\mathbf{x} = t\mathbf{v}$. If we let $\mathbf{x} = (x, y)$, then this equation can be expressed in vector form as

$$(x, y) = t(-2, 3)$$

Equating corresponding components on the two sides of this equation yields the parametric equations

$$x = -2t, \quad y = 3t$$

Solution (b) It follows from (5) that a vector equation of the line is $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$. If we let $\mathbf{x} = (x, y, z)$, and if we take $\mathbf{x}_0 = (1, 2, -3)$, then this equation can be expressed in vector form as

$$(x, y, z) = (1, 2, -3) + t(4, -5, 1) \tag{7}$$

Equating corresponding components on the two sides of this equation yields the parametric equations

$$x = 1 + 4t, \quad y = 2 - 5t, \quad z = -3 + t$$

Solution (c) A point on the line represented by Equation (7) can be obtained by substituting a specific numerical value for the parameter t. However, since $t = 0$ produces $(x, y, z) = (1, 2, -3)$, which is the point P_0, this value of t does not serve our purpose. Taking $t = 1$ produces the point $(5, -3, -2)$ and taking $t = -1$ produces the point $(-3, 7, -4)$. Any other distinct values for t (except $t = 0$) would work just as well.

▶ EXAMPLE 2 Vector and Parametric Equations of a Plane in R^3

Find vector and parametric equations of the plane $x - y + 2z = 5$.

Solution We will find the parametric equations first. We can do this by solving the equation for any one of the variables in terms of the other two and then using those two variables as parameters. For example, solving for x in terms of y and z yields

$$x = 5 + y - 2z \tag{8}$$

and then using y and z as parameters t_1 and t_2, respectively, yields the parametric equations

$$x = 5 + t_1 - 2t_2, \quad y = t_1, \quad z = t_2$$

To obtain a vector equation of the plane we rewrite these parametric equations as

$$(x, y, z) = (5 + t_1 - 2t_2, t_1, t_2)$$

or, equivalently, as

$$(x, y, z) = (5, 0, 0) + t_1(1, 1, 0) + t_2(-2, 0, 1)$$

> We would have obtained different parametric and vector equations in Example 2 had we solved (8) for y or z rather than x. However, one can show the same plane results in all three cases as the parameters vary from $-\infty$ to ∞.

▶ EXAMPLE 3 Vector and Parametric Equations of Lines and Planes in R^4

(a) Find vector and parametric equations of the line through the origin of R^4 that is parallel to the vector $\mathbf{v} = (5, -3, 6, 1)$.

(b) Find vector and parametric equations of the plane in R^4 that passes through the point $\mathbf{x}_0 = (2, -1, 0, 3)$ and is parallel to both $\mathbf{v}_1 = (1, 5, 2, -4)$ and $\mathbf{v}_2 = (0, 7, -8, 6)$.

Solution (a) If we let $\mathbf{x} = (x_1, x_2, x_3, x_4)$, then the vector equation $\mathbf{x} = t\mathbf{v}$ can be expressed as

$$(x_1, x_2, x_3, x_4) = t(5, -3, 6, 1)$$

Equating corresponding components yields the parametric equations

$$x_1 = 5t, \quad x_2 = -3t, \quad x_3 = 6t, \quad x_4 = t$$

Solution (b) The vector equation $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ can be expressed as

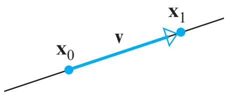
$$(x_1, x_2, x_3, x_4) = (2, -1, 0, 3) + t_1(1, 5, 2, -4) + t_2(0, 7, -8, 6)$$

which yields the parametric equations

$$\begin{aligned} x_1 &= 2 + t_1 \\ x_2 &= -1 + 5t_1 + 7t_2 \\ x_3 &= 2t_1 - 8t_2 \\ x_4 &= 3 - 4t_1 + 6t_2 \end{aligned}$$ ◀

Lines Through Two Points in R^n

If $\mathbf{x}_0$ and $\mathbf{x}_1$ are distinct points in R^n, then the line determined by these points is parallel to the vector $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$ (Figure 3.4.5), so it follows from (5) that the line can be expressed in vector form as



▲ Figure 3.4.5

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \tag{9}$$

or, equivalently, as

$$\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1 \tag{10}$$

These are called the ***two-point vector equations*** of a line in R^n.

▶ EXAMPLE 4 A Line Through Two Points in R^2

Find vector and parametric equations for the line in R^2 that passes through the points $P(0, 7)$ and $Q(5, 0)$.

Solution We will see below that it does not matter which point we take to be $\mathbf{x}_0$ and which we take to be $\mathbf{x}_1$, so let us choose $\mathbf{x}_0 = (0, 7)$ and $\mathbf{x}_1 = (5, 0)$. It follows that $\mathbf{x}_1 - \mathbf{x}_0 = (5, -7)$ and hence that

$$(x, y) = (0, 7) + t(5, -7) \tag{11}$$

which we can rewrite in parametric form as

$$x = 5t, \quad y = 7 - 7t$$

Had we reversed our choices and taken $\mathbf{x}_0 = (5, 0)$ and $\mathbf{x}_1 = (0, 7)$, then the resulting vector equation would have been

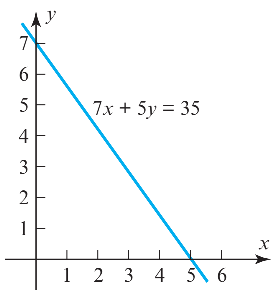
$$(x, y) = (5, 0) + t(-5, 7) \tag{12}$$

and the parametric equations would have been

$$x = 5 - 5t, \quad y = 7t$$

(verify). Although (11) and (12) look different, they both represent the line whose equation in rectangular coordinates is

$$7x + 5y = 35$$



▲ Figure 3.4.6

(Figure 3.4.6). This can be seen by eliminating the parameter t from the parametric equations (verify). ◀

The point $\mathbf{x} = (x, y)$ in Equations (9) and (10) traces an entire line in R^2 as the parameter t varies over the interval $(-\infty, \infty)$. If, however, we restrict the parameter to vary from $t = 0$ to $t = 1$, then $\mathbf{x}$ will not trace the entire line but rather just the *line segment* joining the points $\mathbf{x}_0$ and $\mathbf{x}_1$. The point $\mathbf{x}$ will start at $\mathbf{x}_0$ when $t = 0$ and end at $\mathbf{x}_1$ when $t = 1$. Accordingly, we make the following definition.

DEFINITION 3 If $\mathbf{x}_0$ and $\mathbf{x}_1$ are vectors in R^n, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \quad (0 \le t \le 1) \tag{13}$$

defines the ***line segment from*** $\mathbf{x}_0$ ***to*** $\mathbf{x}_1$. When convenient, Equation (13) can be written as

$$\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1 \quad (0 \le t \le 1) \tag{14}$$

EXAMPLE 5 A Line Segment from One Point to Another in R^2

It follows from (13) and (14) that the line segment in R^2 from $\mathbf{x}_0 = (1, -3)$ to $\mathbf{x}_1 = (5, 6)$ can be represented either by the equation

$$\mathbf{x} = (1, -3) + t(4, 9) \quad (0 \le t \le 1)$$

or by the equation

$$\mathbf{x} = (1 - t)(1, -3) + t(5, 6) \quad (0 \le t \le 1)$$

Dot Product Form of a Linear System

Our next objective is to show how to express linear equations and linear systems in dot product notation. This will lead us to some important results about orthogonality and linear systems.

Recall that a *linear equation* in the variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (a_1, a_2, \ldots, a_n \text{ not all zero}) \tag{15}$$

and that the corresponding *homogeneous* equation is

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \quad (a_1, a_2, \ldots, a_n \text{ not all zero}) \tag{16}$$

These equations can be rewritten in vector form by letting

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)$$

in which case Formula (15) can be written as

$$\mathbf{a} \cdot \mathbf{x} = b \tag{17}$$

and Formula (16) as

$$\mathbf{a} \cdot \mathbf{x} = 0 \tag{18}$$

Except for a notational change from $\mathbf{n}$ to $\mathbf{a}$, Formula (18) is the extension to R^n of Formula (6) in Section 3.3. This equation reveals that *each solution vector* $\mathbf{x}$ *of a homogeneous equation is orthogonal to the coefficient vector* $\mathbf{a}$. To take this geometric observation a step further, consider the homogeneous system

$$\begin{array}{ccccccc} a_{11}x_1 & + a_{12}x_2 & + \cdots + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots + & a_{2n}x_n & = 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + \cdots + & a_{mn}x_n & = 0 \end{array}$$

If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$, then we can rewrite this system in dot product form as

$$\begin{array}{c} \mathbf{r}_1 \cdot \mathbf{x} = \mathbf{0} \\ \mathbf{r}_2 \cdot \mathbf{x} = \mathbf{0} \\ \vdots \quad\quad \vdots \\ \mathbf{r}_m \cdot \mathbf{x} = \mathbf{0} \end{array} \tag{19}$$

from which we see that every solution vector $\mathbf{x}$ is orthogonal to every row vector of the coefficient matrix. In summary, we have the following result.

THEOREM 3.4.3 *If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.*

▶ EXAMPLE 6 Orthogonality of Row Vectors and Solution Vectors

We showed in Example 6 of Section 1.2 that the general solution of the homogeneous linear system

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which we can rewrite in vector form as

$$\mathbf{x} = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

According to Theorem 3.4.3, the vector $\mathbf{x}$ must be orthogonal to each of the row vectors

$$\begin{aligned} \mathbf{r}_1 &= (1, 3, -2, 0, 2, 0) \\ \mathbf{r}_2 &= (2, 6, -5, -2, 4, -3) \\ \mathbf{r}_3 &= (0, 0, 5, 10, 0, 15) \\ \mathbf{r}_4 &= (2, 6, 0, 8, 4, 18) \end{aligned}$$

We will confirm that $\mathbf{x}$ is orthogonal to $\mathbf{r}_1$, and leave it for you to verify that $\mathbf{x}$ is orthogonal to the other three row vectors as well. The dot product of $\mathbf{r}_1$ and $\mathbf{x}$ is

$$\mathbf{r}_1 \cdot \mathbf{x} = 1(-3r - 4s - 2t) + 3(r) + (-2)(-2s) + 0(s) + 2(t) + 0(0) = 0$$

which establishes the orthogonality. ◀

The Relationship Between $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

We will conclude this section by exploring the relationship between the solutions of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ and the solutions (if any) of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ that has the same coefficient matrix. These are called ***corresponding linear systems***.

To motivate the result we are seeking, let us compare the solutions of the corresponding linear systems

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

We showed in Examples 5 and 6 of Section 1.2 that the general solutions of these linear systems can be written in parametric form as

homogeneous $\longrightarrow x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$

nonhomogeneous $\longrightarrow x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$

which we can then rewrite in vector form as

homogeneous $\longrightarrow$ $(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$
nonhomogeneous $\longrightarrow$ $(x_1, x_2, x_3, x_4, x_5, x_6) = \left(-3r - 4s - 2t, r, -2s, s, t, \frac{1}{3}\right)$

By splitting the vectors on the right apart and collecting terms with like parameters, we can rewrite these equations as

homogeneous $\longrightarrow$ $$(x_1, x_2, x_3, x_4, x_5) = r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0) \quad (20)$$

nonhomogeneous $\longrightarrow$ $$(x_1, x_2, x_3, x_4, x_5) = r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0) + \left(0, 0, 0, 0, 0, \frac{1}{3}\right) \quad (21)$$

Formulas (20) and (21) reveal that each solution of the nonhomogeneous system can be obtained by adding the fixed vector $\left(0, 0, 0, 0, 0, \frac{1}{3}\right)$ to the corresponding solution of the homogeneous system. This is a special case of the following general result.

THEOREM 3.4.4 *The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.*

Proof Let $\mathbf{x}_0$ be any specific solution of $A\mathbf{x} = \mathbf{b}$, let W denote the solution set of $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{x}_0 + W$ denote the set of all vectors that result by adding $\mathbf{x}_0$ to each vector in W. We must show that if $\mathbf{x}$ is a vector in $\mathbf{x}_0 + W$, then $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$, and conversely that every solution of $A\mathbf{x} = \mathbf{b}$ is in the set $\mathbf{x}_0 + W$.

Assume first that $\mathbf{x}$ is a vector in $\mathbf{x}_0 + W$. This implies that $\mathbf{x}$ is expressible in the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{w}$, where $A\mathbf{x}_0 = \mathbf{b}$ and $A\mathbf{w} = \mathbf{0}$. Thus,

$$A\mathbf{x} = A(\mathbf{x}_0 + \mathbf{w}) = A\mathbf{x}_0 + A\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

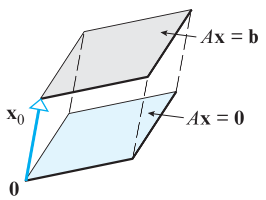
which shows that $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Conversely, let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. To show that $\mathbf{x}$ is in the set $\mathbf{x}_0 + W$ we must show that $\mathbf{x}$ is expressible in the form

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{w} \quad (22)$$

where $\mathbf{w}$ is in W (i.e., $A\mathbf{w} = \mathbf{0}$). We can do this by taking $\mathbf{w} = \mathbf{x} - \mathbf{x}_0$. This vector obviously satisfies (22), and it is in W since

$$A\mathbf{w} = A(\mathbf{x} - \mathbf{x}_0) = A\mathbf{x} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0} \blacktriangleleft$$



▲ Figure 3.4.7 The solution set of $A\mathbf{x} = \mathbf{b}$ is a translation of the solution space of $A\mathbf{x} = \mathbf{0}$.

Remark Theorem 3.4.4 has a useful geometric interpretation that is illustrated in Figure 3.4.7. If, as discussed in Section 3.1, we interpret vector addition as translation, then the theorem states that if $\mathbf{x}_0$ is *any* specific solution of $A\mathbf{x} = \mathbf{b}$, then the *entire* solution set of $A\mathbf{x} = \mathbf{b}$ can be obtained by translating the solution space of $A\mathbf{x} = \mathbf{0}$ by the vector $\mathbf{x}_0$.

Exercise Set 3.4

▶ In Exercises 1–4, find vector and parametric equations of the line containing the point and parallel to the vector. ◀

1. Point: $(-4, 1)$; vector: $\mathbf{v} = (0, -8)$
2. Point: $(2, -1)$; vector: $\mathbf{v} = (-4, -2)$
3. Point: $(0, 0, 0)$; vector: $\mathbf{v} = (-3, 0, 1)$
4. Point: $(-9, 3, 4)$; vector: $\mathbf{v} = (-1, 6, 0)$

▶ In Exercises 5–8, use the given equation of a line to find a point on the line and a vector parallel to the line. ◀

5. $\mathbf{x} = (3 - 5t, -6 - t)$
6. $(x, y, z) = (4t, 7, 4 + 3t)$
7. $\mathbf{x} = (1 - t)(4, 6) + t(-2, 0)$
8. $\mathbf{x} = (1 - t)(0, -5, 1)$

▶ In Exercises 9–12, find vector and parametric equations of the plane that contains the given point and is parallel to the two vectors. ◀

9. Point: $(-3, 1, 0)$; vectors: $\mathbf{v}_1 = (0, -3, 6)$ and $\mathbf{v}_2 = (-5, 1, 2)$

10. Point: $(0, 6, -2)$; vectors: $\mathbf{v}_1 = (0, 9, -1)$ and $\mathbf{v}_2 = (0, -3, 0)$

11. Point: $(-1, 1, 4)$; vectors: $\mathbf{v}_1 = (6, -1, 0)$ and $\mathbf{v}_2 = (-1, 3, 1)$

12. Point: $(0, 5, -4)$; vectors: $\mathbf{v}_1 = (0, 0, -5)$ and $\mathbf{v}_2 = (1, -3, -2)$

▶ In Exercises 13–14, find vector and parametric equations of the line in R^2 that passes through the origin and is orthogonal to $\mathbf{v}$. ◀

13. $\mathbf{v} = (-2, 3)$

14. $\mathbf{v} = (1, -4)$

▶ In Exercises 15–16, find vector and parametric equations of the plane in R^3 that passes through the origin and is orthogonal to $\mathbf{v}$. ◀

15. $\mathbf{v} = (4, 0, -5)$ [*Hint:* Construct two nonparallel vectors orthogonal to $\mathbf{v}$ in R^3].

16. $\mathbf{v} = (3, 1, -6)$

▶ In Exercises 17–20, find the general solution to the linear system and confirm that the row vectors of the coefficient matrix are orthogonal to the solution vectors. ◀

17.
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 3x_3 &= 0 \end{aligned}$$

18.
$$\begin{aligned} x_1 + 3x_2 - 4x_3 &= 0 \\ 2x_1 + 6x_2 - 8x_3 &= 0 \end{aligned}$$

19.
$$\begin{aligned} x_1 + 5x_2 + x_3 + 2x_4 - x_5 &= 0 \\ x_1 - 2x_2 - x_3 + 3x_4 + 2x_5 &= 0 \end{aligned}$$

20.
$$\begin{aligned} x_1 + 3x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned}$$

21. (a) The equation $x + y + z = 1$ can be viewed as a linear system of one equation in three unknowns. Express a general solution of this equation as a particular solution plus a general solution of the associated homogeneous equation.

(b) Give a geometric interpretation of the result in part (a).

22. (a) The equation $x + y = 1$ can be viewed as a linear system of one equation in two unknowns. Express a general solution of this equation as a particular solution plus a general solution of the associated homogeneous system.

(b) Give a geometric interpretation of the result in part (a).

23. (a) Find a homogeneous linear system of two equations in three unknowns whose solution space consists of those vectors in R^3 that are orthogonal to $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (-2, 3, 0)$.

(b) What kind of geometric object is the solution space?

(c) Find a general solution of the system obtained in part (a), and confirm that Theorem 3.4.3 holds.

24. (a) Find a homogeneous linear system of two equations in three unknowns whose solution space consists of those vectors in R^3 that are orthogonal to $\mathbf{a} = (-3, 2, -1)$ and $\mathbf{b} = (0, -2, -2)$.

(b) What kind of geometric object is the solution space?

(c) Find a general solution of the system obtained in part (a), and confirm that Theorem 3.4.3 holds.

25. Consider the linear systems

$$\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

(a) Find a general solution of the homogeneous system.

(b) Confirm that $x_1 = 1, x_2 = 0, x_3 = 1$ is a solution of the nonhomogeneous system.

(c) Use the results in parts (a) and (b) to find a general solution of the nonhomogeneous system.

(d) Check your result in part (c) by solving the nonhomogeneous system directly.

26. Consider the linear systems

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$$

(a) Find a general solution of the homogeneous system.

(b) Confirm that $x_1 = 1, x_2 = 1, x_3 = 1$ is a solution of the nonhomogeneous system.

(c) Use the results in parts (a) and (b) to find a general solution of the nonhomogeneous system.

(d) Check your result in part (c) by solving the nonhomogeneous system directly.

▶ In Exercises 27–28, find a general solution of the system, and use that solution to find a general solution of the associated homogeneous system and a particular solution of the given system. ◀

27.
$$\begin{bmatrix} 3 & 4 & 1 & 2 \\ 6 & 8 & 2 & 5 \\ 9 & 12 & 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}$$

28. $\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$

29. Let $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ be a line in R^n, and let $T: R^n \to R^n$ be a matrix operator on R^n. What kind of geometric object is the image of this line under the operator T? Explain your reasoning.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) The vector equation of a line can be determined from any point lying on the line and a nonzero vector parallel to the line.

(b) The vector equation of a plane can be determined from any point lying in the plane and a nonzero vector parallel to the plane.

(c) The points lying on a line through the origin in R^2 or R^3 are all scalar multiples of any nonzero vector on the line.

(d) All solution vectors of the linear system $A\mathbf{x} = \mathbf{b}$ are orthogonal to the row vectors of the matrix A if and only if $\mathbf{b} = \mathbf{0}$.

(e) The general solution of the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding $\mathbf{b}$ to the general solution of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(f) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution of the corresponding homogeneous linear system.

Working with Technology

T1. Find the general solution of the homogeneous linear system

$$\begin{bmatrix} 2 & 6 & -4 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 6 & 18 & -15 & -6 & 12 & -9 \\ 1 & 3 & 0 & 4 & 2 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and confirm that each solution vector is orthogonal to every row vector of the coefficient matrix in accordance with Theorem 3.4.3.

3.5 Cross Product

This optional section is concerned with properties of vectors in 3-space that are important to physicists and engineers. It can be omitted, if desired, since subsequent sections do not depend on its content. Among other things, we define an operation that provides a way of constructing a vector in 3-space that is perpendicular to two given vectors, and we give a geometric interpretation of 3×3 determinants.

Cross Product of Vectors

In Section 3.2 we defined the dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in n-space. That operation produced a *scalar* as its result. We will now define a type of vector multiplication that produces a *vector* as the result but which is applicable only to vectors in 3-space.

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{1}$$

Remark Instead of memorizing (1), you can obtain the components of $\mathbf{u} \times \mathbf{v}$ as follows:

- Form the 2×3 matrix $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$ whose first row contains the components of $\mathbf{u}$ and whose second row contains the components of $\mathbf{v}$.

- To find the first component of $\mathbf{u} \times \mathbf{v}$, delete the first column and take the determinant; to find the second component, delete the second column and take the negative of the determinant; and to find the third component, delete the third column and take the determinant.

▶ EXAMPLE 1 Calculating a Cross Product

Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$.

Solution From either (1) or the mnemonic in the preceding remark, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right) \\ &= (2, -7, -6) \end{aligned}$$ ◀

The following theorem gives some important relationships between the dot product and cross product and also shows that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

The formulas for the vector triple products in parts (*d*) and (*e*) of Theorem 3.5.1 are useful because they allow us to use dot products and scalar multiplications to perform calculations that would otherwise require determinants to calculate the required cross products.

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* 3-*space*, *then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$]

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$]

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ [*Lagrange's identity*]

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ [*vector triple product*]

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ [*vector triple product*]

Proof (a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_1, u_2, u_3) \cdot (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \\ &= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0 \end{aligned}$$

Proof (b) Similar to (*a*).

Proof (c) Since

$$\|\mathbf{u} \times \mathbf{v}\|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \tag{2}$$

and

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \tag{3}$$

the proof can be completed by "multiplying out" the right sides of (2) and (3) and verifying their equality.

Proof (d) and (e) See Exercises 40 and 41. ◀

Historical Note The cross product notation $A \times B$ was introduced by the American physicist and mathematician J. Willard Gibbs, (see p. 146) in a series of unpublished lecture notes for his students at Yale University. It appeared in a published work for the first time in the second edition of the book *Vector Analysis*, by Edwin Wilson (1879–1964), a student of Gibbs. Gibbs originally referred to $A \times B$ as the "skew product."

EXAMPLE 2 u × v Is Perpendicular to u and to v

Consider the vectors

$$\mathbf{u} = (1, 2, -2) \quad \text{and} \quad \mathbf{v} = (3, 0, 1)$$

In Example 1 we showed that

$$\mathbf{u} \times \mathbf{v} = (2, -7, -6)$$

Since

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (1)(2) + (2)(-7) + (-2)(-6) = 0$$

and

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (3)(2) + (0)(-7) + (1)(-6) = 0$$

$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, as guaranteed by Theorem 3.5.1. ◀

The main arithmetic properties of the cross product are listed in the next theorem.

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in* 3*-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

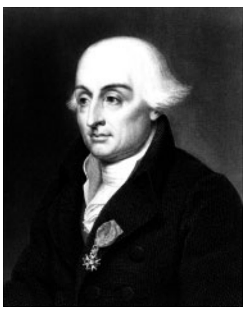
(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

The proofs follow immediately from Formula (1) and properties of determinants; for example, part (*a*) can be proved as follows.

Proof (a) Interchanging $\mathbf{u}$ and $\mathbf{v}$ in (1) interchanges the rows of the three determinants on the right side of (1) and hence changes the sign of each component in the cross product. Thus $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. ◀

The proofs of the remaining parts are left as exercises.



Joseph Louis Lagrange (1736–1813)

Historical Note Joseph Louis Lagrange was a French-Italian mathematician and astronomer. Although his father wanted him to become a lawyer, Lagrange was attracted to mathematics and astronomy after reading a memoir by the astronomer Halley. At age 16 he began to study mathematics on his own and by age 19 was appointed to a professorship at the Royal Artillery School in Turin. The following year he solved some famous problems using new methods that eventually blossomed into a branch of mathematics called the *calculus of variations*. These methods and Lagrange's applications of them to problems in celestial mechanics were so monumental that by age 25 he was regarded by many of his contemporaries as the greatest living mathematician. One of Lagrange's most famous works is a memoir, *Mécanique Analytique*, in which he reduced the theory of mechanics to a few general formulas from which all other necessary equations could be derived. Napoleon was a great admirer of Lagrange and showered him with many honors. In spite of his fame, Lagrange was a shy and modest man. On his death, he was buried with honor in the Pantheon.

EXAMPLE 3 Cross Products of the Standard Unit Vectors

Recall from Section 3.2 that the standard unit vectors in 3-space are

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

These vectors each have length 1 and lie along the coordinate axes (Figure 3.5.1). Every vector $\mathbf{v} = (v_1, v_2, v_3)$ in 3-space is expressible in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ since we can write

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

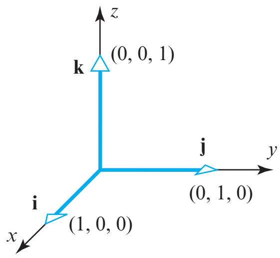


Figure 3.5.1 The standard unit vectors.

For example,

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

From (1) we obtain

$$\mathbf{i} \times \mathbf{j} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (0, 0, 1) = \mathbf{k}$$

You should have no trouble obtaining the following results:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

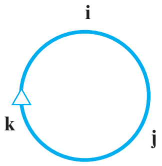


Figure 3.5.2

Figure 3.5.2 is helpful for remembering these results. Referring to this diagram, the cross product of two consecutive vectors going clockwise is the next vector around, and the cross product of two consecutive vectors going counterclockwise is the negative of the next vector around.

Determinant Form of Cross Product

It is also worth noting that a cross product can be represented symbolically in the form

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \tag{4}$$

For example, if $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$$

which agrees with the result obtained in Example 1.

WARNING It is not true in general that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. For example,

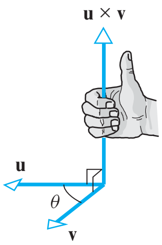
$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$$

and

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

so

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$$



▲ Figure 3.5.3

We know from Theorem 3.5.1 that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, it can be shown that the direction of $\mathbf{u} \times \mathbf{v}$ can be determined using the following "right-hand rule" (Figure 3.5.3): Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and suppose $\mathbf{u}$ is rotated through the angle θ until it coincides with $\mathbf{v}$. If the fingers of the right hand are cupped so that they point in the direction of rotation, then the thumb indicates (roughly) the direction of $\mathbf{u} \times \mathbf{v}$.

You may find it instructive to practice this rule with the products

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Geometric Interpretation of Cross Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, then the norm of $\mathbf{u} \times \mathbf{v}$ has a useful geometric interpretation. Lagrange's identity, given in Theorem 3.5.1, states that

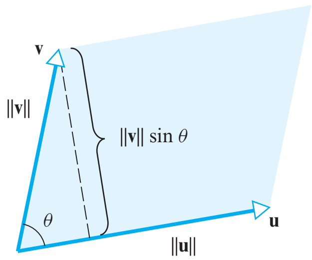
$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \tag{5}$$

If θ denotes the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, so (5) can be rewritten as

$$\begin{aligned}\|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \cos^2\theta \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1 - \cos^2\theta) \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta\end{aligned}$$

Since $0 \le \theta \le \pi$, it follows that $\sin\theta \ge 0$, so this can be rewritten as

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta \tag{6}$$



▲ Figure 3.5.4

But $\|\mathbf{v}\| \sin\theta$ is the altitude of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ (Figure 3.5.4). Thus, from (6), the area A of this parallelogram is given by

$$A = \text{(base)(altitude)} = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|$$

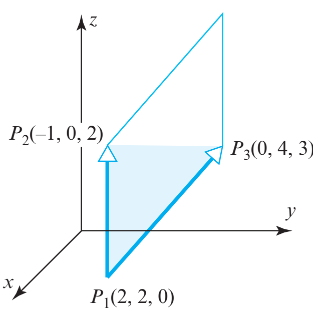
This result is even correct if $\mathbf{u}$ and $\mathbf{v}$ are collinear, since the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ has zero area and from (6) we have $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ because $\theta = 0$ in this case. Thus we have the following theorem.

THEOREM 3.5.3 Area of a Parallelogram

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, then $\|\mathbf{u} \times \mathbf{v}\|$ is equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

▶ EXAMPLE 4 Area of a Triangle

Find the area of the triangle determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.



▲ Figure 3.5.5

Solution The area A of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ (Figure 3.5.5). Using the method discussed in Example 1 of Section 3.1, $\overrightarrow{P_1P_2} = (-3, -2, 2)$ and $\overrightarrow{P_1P_3} = (-2, 2, 3)$. It follows that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-10, 5, -10)$$

(verify) and consequently that

$$A = \tfrac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \tfrac{1}{2}(15) = \tfrac{15}{2} \quad ◀$$

DEFINITION 2 If **u**, **v**, and **w** are vectors in 3-space, then

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$$

is called the ***scalar triple product*** of **u**, **v**, and **w**.

The scalar triple product of $\mathbf{u}=(u_1,u_2,u_3)$, $\mathbf{v}=(v_1,v_2,v_3)$, and $\mathbf{w}=(w_1,w_2,w_3)$ can be calculated from the formula

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{7}$$

This follows from Formula (4) since

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \mathbf{u}\cdot\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k}\right)\\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}u_1-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}u_2+\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}u_3\\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\end{aligned}$$

▶ EXAMPLE 5 Calculating a Scalar Triple Product

Calculate the scalar triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ of the vectors

$$\mathbf{u}=3\mathbf{i}-2\mathbf{j}-5\mathbf{k},\quad \mathbf{v}=\mathbf{i}+4\mathbf{j}-4\mathbf{k},\quad \mathbf{w}=3\mathbf{j}+2\mathbf{k}$$

Solution From (7),

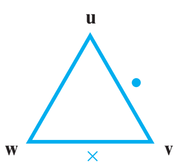
$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}\\ &= 3\begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix}-(-2)\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}+(-5)\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}\\ &= 60+4-15=49\end{aligned}$$ ◀

Remark The symbol $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$ makes no sense because we cannot form the cross product of a scalar and a vector. Thus, no ambiguity arises if we write $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$ rather than $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. However, for clarity we will usually keep the parentheses.

It follows from (7) that

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$$

since the 3×3 determinants that represent these products can be obtained from one another by *two* row interchanges. (Verify.) These relationships can be remembered by moving the vectors **u**, **v**, and **w** clockwise around the vertices of the triangle in Figure 3.5.6.



▲ Figure 3.5.6

Geometric Interpretation of Determinants

The next theorem provides a useful geometric interpretation of 2×2 and 3×3 determinants.

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

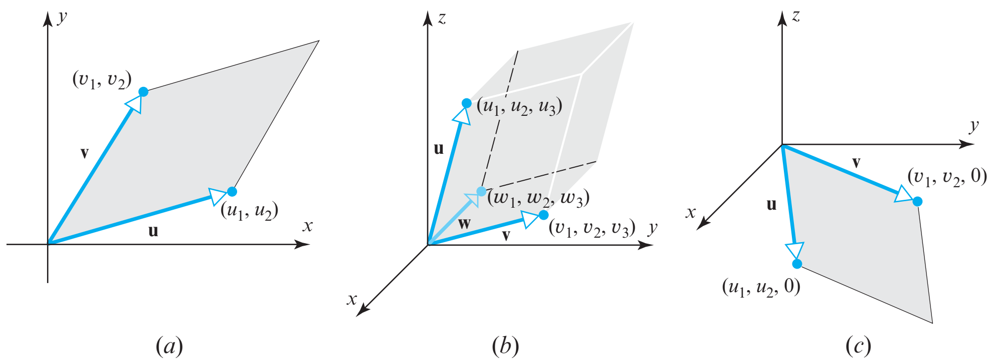
$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*



▲ Figure 3.5.7

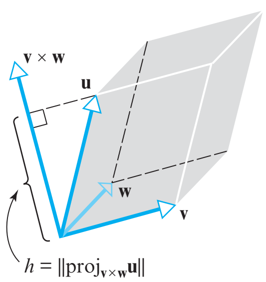
Proof (a) The key to the proof is to use Theorem 3.5.3. However, that theorem applies to vectors in 3-space, whereas $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in 2-space. To circumvent this "dimension problem," we will view $\mathbf{u}$ and $\mathbf{v}$ as vectors in the xy-plane of an xyz-coordinate system (Figure 3.5.7c), in which case these vectors are expressed as $\mathbf{u} = (u_1, u_2, 0)$ and $\mathbf{v} = (v_1, v_2, 0)$. Thus

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} = \det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\mathbf{k}$$

It now follows from Theorem 3.5.3 and the fact that $\|\mathbf{k}\| = 1$ that the area A of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is

$$A = \|\mathbf{u} \times \mathbf{v}\| = \left\|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\mathbf{k}\right\| = \left|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right| \|\mathbf{k}\| = \left|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|$$

which completes the proof.



▲ Figure 3.5.8

Proof (b) As shown in Figure 3.5.8, take the base of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ to be the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. It follows from Theorem 3.5.3 that the area of the base is $\|\mathbf{v} \times \mathbf{w}\|$ and, as illustrated in Figure 3.5.8, the height h of the parallelepiped is the length of the orthogonal projection of $\mathbf{u}$ on $\mathbf{v} \times \mathbf{w}$. Therefore, by Formula (12) of Section 3.3,

$$h = \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}$$

It follows that the volume V of the parallelepiped is

$$V = \text{(area of base)}\cdot\text{height} = \|\mathbf{v}\times\mathbf{w}\|\frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|} = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$$

so from (7),

$$V = \left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right| \tag{8}$$

which completes the proof. ◀

Remark If V denotes the volume of the parallelepiped determined by vectors **u**, **v**, and **w**, then it follows from Formulas (7) and (8) that

$$V = \begin{bmatrix} \text{volume of parallelepiped} \\ \text{determined by } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \end{bmatrix} = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| \tag{9}$$

From this result and the discussion immediately following Definition 3 of Section 3.2, we can conclude that

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \pm V$$

where the + or − results depending on whether **u** makes an acute or an obtuse angle with $\mathbf{v}\times\mathbf{w}$.

Formula (9) leads to a useful test for ascertaining whether three given vectors lie in the same plane. Since three vectors not in the same plane determine a parallelepiped of positive volume, it follows from (9) that $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = 0$ if and only if the vectors **u**, **v**, and **w** lie in the same plane. Thus we have the following result.

THEOREM 3.5.5 *If the vectors* $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$ *have the same initial point, then they lie in the same plane if and only if*

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Exercise Set 3.5

▶ In Exercises 1–2, let $\mathbf{u} = (3, 2, -1)$, $\mathbf{v} = (0, 2, -3)$, and $\mathbf{w} = (2, 6, 7)$. Compute the indicated vectors. ◀

1. (a) $\mathbf{v}\times\mathbf{w}$ (b) $\mathbf{w}\times\mathbf{v}$ (c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}$

(d) $\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w})$ (e) $\mathbf{v}\times\mathbf{v}$ (f) $(\mathbf{u}-3\mathbf{w})\times(\mathbf{u}-3\mathbf{w})$

2. (a) $\mathbf{u}\times\mathbf{v}$ (b) $-(\mathbf{u}\times\mathbf{v})$ (c) $\mathbf{u}\times(\mathbf{v}+\mathbf{w})$

(d) $\mathbf{w}\cdot(\mathbf{w}\times\mathbf{v})$ (e) $\mathbf{w}\times\mathbf{w}$ (f) $(7\mathbf{v}-3\mathbf{u})\times(7\mathbf{v}-3\mathbf{u})$

▶ In Exercises 3–4, let **u**, **v**, and **w** be the vectors in Exercises 1–2. Use Lagrange's identity to rewrite the expression using only dot products and scalar multiplications, and then confirm your result by evaluating both sides of the identity. ◀

3. $\|\mathbf{u}\times\mathbf{w}\|^2$ **4.** $\|\mathbf{v}\times\mathbf{u}\|^2$

▶ In Exercises 5–6, let **u**, **v**, and **w** be the vectors in Exercises 1–2. Compute the vector triple product directly, and check your result by using parts (*d*) and (*e*) of Theorem 3.5.1. ◀

5. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ **6.** $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$

▶ In Exercises 7–8, use the cross product to find a vector that is orthogonal to both **u** and **v**. ◀

7. $\mathbf{u} = (-6, 4, 2)$, $\mathbf{v} = (3, 1, 5)$

8. $\mathbf{u} = (1, 1, -2)$, $\mathbf{v} = (2, -1, 2)$

▶ In Exercises 9–10, find the area of the parallelogram determined by the given vectors **u** and **v**. ◀

9. $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (0, 3, 1)$

10. $\mathbf{u} = (3, -1, 4)$, $\mathbf{v} = (6, -2, 8)$

▶ In Exercises 11–12, find the area of the parallelogram with the given vertices. ◀

11. $P_1(1, 2)$, $P_2(4, 4)$, $P_3(7, 5)$, $P_4(4, 3)$

12. $P_1(3, 2)$, $P_2(5, 4)$, $P_3(9, 4)$, $P_4(7, 2)$

▶ In Exercises 13–14, find the area of the triangle with the given vertices. ◀

13. $A(2, 0)$, $B(3, 4)$, $C(-1, 2)$

14. $A(1, 1)$, $B(2, 2)$, $C(3, -3)$

▶ In Exercises 15–16, find the area of the triangle in 3-space that has the given vertices. ◀

15. $P_1(2, 6, -1)$, $P_2(1, 1, 1)$, $P_3(4, 6, 2)$

16. $P(1, -1, 2)$, $Q(0, 3, 4)$, $R(6, 1, 8)$

▶ In Exercises 17–18, find the volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. ◀

17. $\mathbf{u} = (2, -6, 2)$, $\mathbf{v} = (0, 4, -2)$, $\mathbf{w} = (2, 2, -4)$

18. $\mathbf{u} = (3, 1, 2)$, $\mathbf{v} = (4, 5, 1)$, $\mathbf{w} = (1, 2, 4)$

▶ In Exercises 19–20, determine whether $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie in the same plane when positioned so that their initial points coincide. ◀

19. $\mathbf{u} = (-1, -2, 1)$, $\mathbf{v} = (3, 0, -2)$, $\mathbf{w} = (5, -4, 0)$

20. $\mathbf{u} = (5, -2, 1)$, $\mathbf{v} = (4, -1, 1)$, $\mathbf{w} = (1, -1, 0)$

▶ In Exercises 21–24, compute the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. ◀

21. $\mathbf{u} = (-2, 0, 6)$, $\mathbf{v} = (1, -3, 1)$, $\mathbf{w} = (-5, -1, 1)$

22. $\mathbf{u} = (-1, 2, 4)$, $\mathbf{v} = (3, 4, -2)$, $\mathbf{w} = (-1, 2, 5)$

23. $\mathbf{u} = (a, 0, 0)$, $\mathbf{v} = (0, b, 0)$, $\mathbf{w} = (0, 0, c)$

24. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{k}$

▶ In Exercises 25–26, suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 3$. Find

25. (a) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ (b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$ (c) $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

26. (a) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ (b) $(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$ (c) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{w})$ ◀

27. (a) Find the area of the triangle having vertices $A(1, 0, 1)$, $B(0, 2, 3)$, and $C(2, 1, 0)$.

(b) Use the result of part (a) to find the length of the altitude from vertex C to side AB.

28. Use the cross product to find the sine of the angle between the vectors $\mathbf{u} = (2, 3, -6)$ and $\mathbf{v} = (2, 3, 6)$.

29. Simplify $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

30. Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$, and $\mathbf{d} = (d_1, d_2, d_3)$. Show that

$$(\mathbf{a} + \mathbf{d}) \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$$

▶ **Exercises 31–32** You know from your own experience that the tendency for a force to cause a rotation about an axis depends on the amount of force applied and its distance from the axis of rotation. For example, it is easier to close a door by pushing on its outer edge than close to its hinges. Moreover, the harder you push, the faster the door will close. In physics, the tendency for a force vector $\mathbf{F}$ to cause rotational motion is a vector called ***torque*** (denoted by $\boldsymbol{\tau}$). It is defined as

$$\boldsymbol{\tau} = \mathbf{F} \times \mathbf{d}$$

where $\mathbf{d}$ is the vector from the axis of rotation to the point at which the force is applied. It follows from Formula (6) that

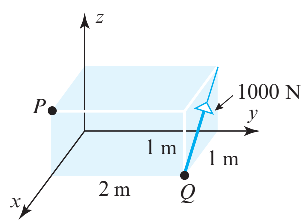
$$\|\boldsymbol{\tau}\| = \|\mathbf{F} \times \mathbf{d}\| = \|\mathbf{F}\|\|\mathbf{d}\| \sin\theta$$

where θ is the angle between the vectors $\mathbf{F}$ and $\mathbf{d}$. This is called the ***scalar moment*** of $\mathbf{F}$ about the axis of rotation and is typically measured in units of Newton-meters (Nm) or foot pounds (ft–lb). ◀

31. The accompanying figure shows a force $\mathbf{F}$ of 1000 N applied to the corner of a box.

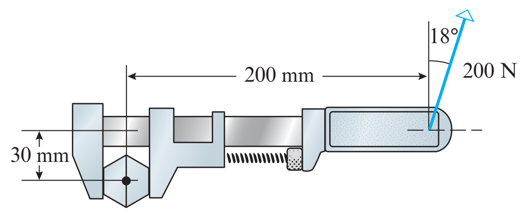
(a) Find the scalar moment of $\mathbf{F}$ about the point P.

(b) Find the direction angles of the vector moment of $\mathbf{F}$ about the point P to the nearest degree. [See directions for Exercises 21–25 of Section 3.2.]



◀ Figure Ex-31

32. As shown in the accompanying figure, a force of 200 N is applied at an angle of 18° to a point near the end of a monkey wrench. Find the scalar moment of the force about the center of the bolt. [*Note:* Treat the wrench as two-dimensional.]



▲ Figure Ex-32

Working with Proofs

33. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in 3-space with the same initial point, but such that no two of them are collinear. Prove that

(a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane determined by $\mathbf{v}$ and $\mathbf{w}$.

(b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$.

34. Prove the following identities.

(a) $(\mathbf{u} + k\mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$

(b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{z}) = -(\mathbf{u} \times \mathbf{z}) \cdot \mathbf{v}$

35. Prove: If **a**, **b**, **c**, and **d** lie in the same plane, then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$.

36. Prove: If θ is the angle between **u** and **v** and $\mathbf{u} \cdot \mathbf{v} \neq 0$, then $\tan\theta = \|\mathbf{u} \times \mathbf{v}\|/(\mathbf{u} \cdot \mathbf{v})$.

37. Prove that if **u**, **v**, and **w** are vectors in R^3, no two of which are collinear, then $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane determined by **v** and **w**.

38. It is a theorem of solid geometry that the volume of a tetrahedron is $\frac{1}{3}$(area of base) · (height). Use this result to prove that the volume of a tetrahedron whose sides are the vectors **a**, **b**, and **c** is $\frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ (see accompanying figure).

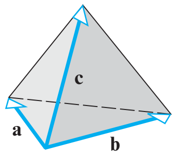


Figure Ex-38

39. Use the result of Exercise 38 to find the volume of the tetrahedron with vertices P, Q, R, S.

(a) $P(-1, 2, 0)$, $Q(2, 1, -3)$, $R(1, 1, 1)$, $S(3, -2, 3)$

(b) $P(0, 0, 0)$, $Q(1, 2, -1)$, $R(3, 4, 0)$, $S(-1, -3, 4)$

40. Prove part (*d*) of Theorem 3.5.1. [*Hint:* First prove the result in the case where $\mathbf{w} = \mathbf{i} = (1, 0, 0)$, then when $\mathbf{w} = \mathbf{j} = (0, 1, 0)$, and then when $\mathbf{w} = \mathbf{k} = (0, 0, 1)$. Finally, prove it for an arbitrary vector $\mathbf{w} = (w_1, w_2, w_3)$ by writing $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$.]

41. Prove part (*e*) of Theorem 3.5.1. [*Hint:* Apply part (*a*) of Theorem 3.5.2 to the result in part (*d*) of Theorem 3.5.1.]

42. Prove:

(a) Prove (*b*) of Theorem 3.5.2.

(b) Prove (*c*) of Theorem 3.5.2.

(c) Prove (*d*) of Theorem 3.5.2.

(d) Prove (*e*) of Theorem 3.5.2.

(e) Prove (*f*) of Theorem 3.5.2.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) The cross product of two nonzero vectors **u** and **v** is a nonzero vector if and only if **u** and **v** are not parallel.

(b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

(c) The scalar triple product of **u**, **v**, and **w** determines a vector whose length is equal to the volume of the parallelepiped determined by **u**, **v**, and **w**.

(d) If **u** and **v** are vectors in 3-space, then $\|\mathbf{v} \times \mathbf{u}\|$ is equal to the area of the parallelogram determined by **u** and **v**.

(e) For all vectors **u**, **v**, and **w** in 3-space, the vectors $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ are the same.

(f) If **u**, **v**, and **w** are vectors in R^3, where **u** is nonzero and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

Working with Technology

T1. As stated in Exercise 23 above, the distance d in 3-space from a point P to the line L through points A and B is given by the formula

$$d = \frac{\|\overrightarrow{AP} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}$$

Find the distance between the point $P(1, 3, 1)$ and the line through the points $A(2, -3, 4)$ and $B(4, 7, -2)$.

Chapter 3 Supplementary Exercises

1. Let $\mathbf{u} = (-2, 0, 4)$, $\mathbf{v} = (3, -1, 6)$, and $\mathbf{w} = (2, -5, -5)$. Compute

(a) $3\mathbf{v} - 2\mathbf{u}$

(b) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$

(c) the distance between $-3\mathbf{u}$ and $\mathbf{v} + 5\mathbf{w}$

(d) $\text{proj}_{\mathbf{w}}\mathbf{u}$

(e) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

(f) $(-5\mathbf{v} + \mathbf{w}) \times ((\mathbf{u} \cdot \mathbf{v})\mathbf{w})$

2. Repeat Exercise 1 for the vectors $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{k}$, and $\mathbf{w} = -\mathbf{j} + 4\mathbf{k}$.

3. Repeat parts (a)–(d) of Exercise 1 for the vectors $\mathbf{u} = (-2, 6, 2, 1)$, $\mathbf{v} = (-3, 0, 8, 0)$, and $\mathbf{w} = (9, 1, -6, -6)$.

4. (a) The set of all vectors in R^2 that are orthogonal to a nonzero vector is what kind of geometric object?

(b) The set of all vectors in R^3 that are orthogonal to a nonzero vector is what kind of geometric object?

(c) The set of all vectors in R^2 that are orthogonal to two noncollinear vectors is what kind of geometric object?

(d) The set of all vectors in R^3 that are orthogonal to two noncollinear vectors is what kind of geometric object?

5. Let A, B, and C be three distinct noncollinear points in 3-space. Describe the set of all points P that satisfy the vector equation $\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$.

6. Let A, B, C, and D be four distinct noncollinear points in 3-space. If $\overrightarrow{AB} \times \overrightarrow{CD} \neq \mathbf{0}$ and $\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CD}) = 0$, explain why the line through A and B must intersect the line through C and D.

7. Consider the points $P(3, -1, 4)$, $Q(6, 0, 2)$, and $R(5, 1, 1)$. Find the point S in R^3 whose first component is -1 and such that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

8. Consider the points $P(-3, 1, 0, 6)$, $Q(0, 5, 1, -2)$, and $R(-4, 1, 4, 0)$. Find the point S in R^4 whose third component is 6 and such that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

9. Using the points in Exercise 7, find the cosine of the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

10. Using the points in Exercise 8, find the cosine of the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

11. Find the distance between the point $P(-3, 1, 3)$ and the plane $5x + z = 3y - 4$.

12. Show that the planes $3x - y + 6z = 7$ and $-6x + 2y - 12z = 1$ are parallel, and find the distance between them.

▶ In Exercises 13–18, find vector and parametric equations for the line or plane in question. ◀

13. The plane in R^3 that contains the points $P(-2, 1, 3)$, $Q(-1, -1, 1)$, and $R(3, 0, -2)$.

14. The line in R^3 that contains the point $P(-1, 6, 0)$ and is orthogonal to the plane $4x - z = 5$.

15. The line in R^2 that is parallel to the vector $\mathbf{v} = (8, -1)$ and contains the point $P(0, -3)$.

16. The plane in R^3 that contains the point $P(-2, 1, 0)$ and is parallel to the plane $-8x + 6y - z = 4$.

17. The line in R^2 with equation $y = 3x - 5$.

18. The plane in R^3 with equation $2x - 6y + 3z = 5$.

▶ In Exercises 19–21, find a point-normal equation for the given plane. ◀

19. The plane that is represented by the vector equation $(x, y, z) = (-1, 5, 6) + t_1(0, -1, 3) + t_2(2, -1, 0)$.

20. The plane that contains the point $P(-5, 1, 0)$ and is orthogonal to the line with parametric equations $x = 3 - 5t$, $y = 2t$, and $z = 7$.

21. The plane that passes through the points $P(9, 0, 4)$, $Q(-1, 4, 3)$, and $R(0, 6, -2)$.

22. Suppose that $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $W = \{\mathbf{w}_1, \mathbf{w}_2\}$ are two sets of vectors such that each vector in V is orthogonal to each vector in W. Prove that if a_1, a_2, a_3, b_1, b_2 are any scalars, then the vectors $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ and $\mathbf{w} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2$ are orthogonal.

23. Show that in 3-space the distance d from a point P to the line L through points A and B can be expressed as

$$d = \frac{\|\overrightarrow{AP} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}$$

24. Prove that $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ if and only if one of the vectors is a scalar multiple of the other.

25. The equation $Ax + By = 0$ represents a line through the origin in R^2 if A and B are not both zero. What does this equation represent in R^3 if you think of it as $Ax + By + 0z = 0$? Explain.